Some results for coupled gradient–type quasilinear elliptic systems with supercritical growth

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- Ω is an open bounded domain in $\mathbb{R}^N$, $N \geq 2$,
- A, $B : \Omega \times \mathbb{R} \to \mathbb{R}$ are $\mathcal{C}^1$ Carathéodory functions, with partial derivatives $A_u(x,u)$, respectively $B_v(x,v)$;

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- $p_1,\ p_2 > 1$;
- a $\mathcal{C}^1$ Carathéodory function $G : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ exists with partial derivatives $G_u(x,u,v)$, $G_v(x,u,v)$.

Model systems

A special case is the classical (p_1, p_2)–Laplacian system:

$$\begin{cases} -\Delta_{p_1} u = G_u(x,u,v) & \text{in } \Omega \\ -\Delta_{p_2} v = G_v(x,u,v) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases}.$$

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An example of the quasilinear system can be written if

$$A(x,u) = 1 + |u|^{s_1 p_1}, \quad B(x,v) = 1 + |v|^{s_2 p_2},$$

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A particular nonlinear term is

$$G(x,u,v) = \frac{1}{q_1}|u|^{q_1} + \frac{1}{q_2}|v|^{q_2} + c_*|u|^{\gamma_1}|v|^{\gamma_2},$$

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If $c_* = 0$: uncoupled system.

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for each $i \in \{1, \ldots, m\}$ a function $A_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ exists which "grows" as $|\xi|^{p_i}$ $(p_i > 1)$ w.r.t. its last N–dimensional variable ξ and is such that

$$A_{i,t}(x,t,\xi) = \frac{\partial A_i}{\partial t}(x,t,\xi),\ \ a_i(x,t,\xi) = \left(\frac{\partial A_i}{\partial \xi_1}(x,t,\xi), \ldots, \frac{\partial A_i}{\partial \xi_N}(x,t,\xi)\right)$$

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and $G : \Omega \times \mathbb{R}^m \to \mathbb{R}$ exists such that

$$G_i(x, \mathbf{u}) = \frac{\partial G}{\partial u_i}(x, \mathbf{u}) \qquad \text{if } 1 \leq i \leq m.$$

About an equation

Also investigating the existence of solutions for just a quasilinear equation

$$\begin{cases} -\mathrm{div}(A(x,u)|\nabla u|^{p-2}\nabla u) + \frac{1}{p}\, A_t(x,u)|\nabla u|^p = \, g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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- a *Newtonian fluid* if $p_i = 2$.

Previous results

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- by using a cohomological local splitting (**A.M.C., E. Medeiros, G. Palmieri and K. Perera** 2010)

First hypotheses on $A(x, u)$, $B(x, u)$, $G(x, u, v)$

Assume that not only $A(x, u)$, $B(x, u)$, $G(x, u, v)$ are $\mathcal{C}^1$ Carathéodory functions but also:

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(h_1) for any $\rho > 0$ we have that

$$\sup_{|u|\leq\rho} |A(\cdot,u)| \in L^\infty(\Omega), \qquad \sup_{|v|\leq\rho} |B(\cdot,v)| \in L^\infty(\Omega),$$

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$G_u(x,0,0) = G_v(x,0,0) = 0$ for a.e. $x \in \Omega$;

(g_1) a constant $\sigma > 0$ and some exponents $q_i \ge 1$, $t_i \ge 0$, if $i \in \{1,2\}$, exist such that

$$|G_u(x,u,v)| \le \sigma(1 + |u|^{q_1-1} + |v|^{t_1}) \text{ for a.e. } x \in \Omega,\ \forall (u,v) \in \mathbb{R}^2,$$

$$|G_v(x,u,v)| \le \sigma(1 + |u|^{t_2} + |v|^{q_2-1}) \text{ for a.e. } x \in \Omega,\ \forall (u,v) \in \mathbb{R}^2.$$

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(h_1), (g_0)–(g_1) $\implies$ $\mathcal{J}$ is well defined in the Banach space

$$X = X_1 \times X_2 = W \cap L, \qquad \|(u,v)\|_X = \|(u,v)\|_W + \|(u,v)\|_L,$$

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- $L = L^\infty(\Omega) \times L^\infty(\Omega)$, $\|(u,v)\|_L = |u|_\infty + |v|_\infty$.

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$$+ \frac{1}{p_2}\int_\Omega B_v(x,v)z|\nabla v|^{p_2} dx - \int_\Omega G_u(x,u,v)w\ dx - \int_\Omega G_v(x,u,v)z\ dx.$$

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For simplicity, we put

$$\frac{\partial \mathcal{J}}{\partial u}(u,v) : w \in X_1 \mapsto \frac{\partial \mathcal{J}}{\partial u}(u,v)[w] = d\mathcal{J}(u,v)[(w,0)] \in \mathbb{R},$$

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Proposition (A.M.C., A. Salvatore, C. Sportelli 2021)

Let $((u_n, v_n))_n \subset X$, $(u, v) \in X$ *and* $M > 0$ *be such that:*

$$(u_n, v_n) \to (u, v) \text{ in } W \quad \text{and} \quad (u_n, v_n) \to (u, v) \text{ a.e. in } \Omega,$$
$$|u_n|_\infty \le M \quad \text{and} \quad |v_n|_\infty \le M \quad \text{for all } n \in \mathbb{N}.$$

Then,

$$\mathcal{J}(u_n, v_n) \to \mathcal{J}(u, v) \quad \text{and} \quad \|d\mathcal{J}(u_n, v_n) - d\mathcal{J}(u, v)\|_{X'} \to 0.$$

Hence, $\mathcal{J}$ *is a* $\mathcal{C}^1$ *functional on* X.

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Both $p_1 > N$ and $p_2 > N \implies W_0^{1,p_i}(\Omega) \hookrightarrow L^\infty(\Omega)$ for both $i = 1$ and $i = 2 \implies X = W$.

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In general, Palais–Smale sequences may converge in $\|(\cdot,\cdot)\|_W$ but not in $\|(\cdot,\cdot)\|_L$ (counterexamples exist).

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Moreover, if J is even then ψ can be chosen odd.

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Then, J has a Mountain Pass critical point $u_0 \in X$ such that $J(u_0) \geq \varrho_0$.

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Then, J possesses at least a pair of symmetric critical points in X with corresponding critical level $\beta_\varrho = \inf_{\gamma \in \Gamma_\varrho} \sup_{u \in V_\varrho} J(\gamma(u))$, *with* $\varrho \leq \beta_\varrho \leq \varrho_1$, *where* $\varrho_1 \geq \sup_{u \in V_\varrho} J(u) > \varrho$.

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Furthermore, if $(\mathcal{H}_\varrho)$ *holds for all* $\varrho > 0$*, then* J *possesses a sequence of critical points* $(u_n)_n \subset X$ *such that* $J(u_n) \nearrow +\infty$.

Back to the coupled gradient–type quasilinear system

Now, assume that (h_1) and (g_0)–(g_1) hold and consider the functional related to the coupled quasilinear system

$$\mathcal{J}(u,v) = \frac{1}{p_1}\int_\Omega A(x,u)|\nabla u|^{p_1}dx + \frac{1}{p_2}\int_\Omega B(x,v)|\nabla v|^{p_2}dx - \int_\Omega G(x,u,v)dx$$

which is a $\mathcal{C}^1$ functional on $X = W \cap L$.

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(h_3) there exist θ_1, θ_2, $\mu_2 > 0$ such that $\theta_1 < \frac{1}{p_1}$, $\theta_2 < \frac{1}{p_2}$,

$$(1 - p_1\theta_1)A(x,u) - \theta_1 A_u(x,u)u \geq \mu_2 A(x,u) \quad \text{a.e. in}\Omega, \ \forall\, u \in \mathbb{R},$$
$$(1 - p_2\theta_2)B(x,v) - \theta_2 B_v(x,v)v \geq \mu_2 B(x,v) \quad \text{a.e. in}\Omega, \ \forall\, v \in \mathbb{R};$$

Some hypotheses

Assume that $R \geq 1$ exists such that the following conditions hold:

(h_2) there exists $\mu_1 > 0$ such that

$$A(x,u) + \frac{1}{p_1} A_u(x,u)u \geq \mu_1 A(x,u) \quad \text{a.e. in } \Omega \text{ if } |u| \geq R,$$

$$B(x,v) + \frac{1}{p_2} B_v(x,v)v \geq \mu_1 B(x,v) \quad \text{a.e. in } \Omega \text{ if } |v| \geq R;$$

(h_3) there exist θ_1, θ_2, $\mu_2 > 0$ such that $\theta_1 < \frac{1}{p_1}$, $\theta_2 < \frac{1}{p_2}$,

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$$(1 - p_2\theta_2)B(x,v) - \theta_2 B_v(x,v)v \geq \mu_2 B(x,v) \quad \text{a.e. in}\Omega,\ \forall\ v \in \mathbb{R};$$

(g_2) $0 < G(x,u,v) \leq \theta_1 G_u(x,u,v)u + \theta_2 G_v(x,u,v)v$ a.e. in Ω if $|(u,v)| \geq R$.

Further hypotheses

(g_3) $\limsup_{(u,v)\to 0} \dfrac{G(x,u,v)}{|u|^{p_1}+|v|^{p_2}} < \mu_0 \min\left\{\dfrac{\lambda_{1,1}}{p_1}, \dfrac{\lambda_{2,1}}{p_2}\right\}$ uniformly a.e. in Ω, with $\lambda_{i,1}$ first eigenvalue of $-\Delta_{p_i}$ in $W_0^{1,p_i}(\Omega)$ if $i \in \{1,2\}$;

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(g_5) $G(x,\cdot,\cdot)$ is even in $\mathbb{R}^2$ for a.e. $x \in \Omega$.

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