Existence of radial bounded solutions for some quasi-linear elliptic equations in $\mathbb{R}^N$

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Outline

The elliptic problem

We look for (weak) solutions of the quasilinear elliptic equation

$$(P) \qquad -\operatorname{div}\left(A(x,u)|\nabla u|^{p-2}\nabla u\right) + \frac{1}{p}A_t(x,u)|\nabla u|^p + |u|^{p-2}u = g(x,u) \qquad \text{in } \mathbb{R}^N$$

where

- $N \geq 3$, $p > 1$;
- $A : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a C^1-Carathéodory function, *i.e.* $A(\cdot,t)$ is measurable for all $t \in \mathbb{R}$ and $A(x,\cdot)$ is C^1 for a.e. $x \in \mathbb{R}^N$ with $A_t(x,t) = \frac{\partial A}{\partial t}(x,t)$;
- $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, *i.e.* $g(\cdot,t)$ is measurable for all $t \in \mathbb{R}$ and $g(x,\cdot)$ is continous for a.e. $x \in \mathbb{R}^N$.

If $A(x,t) \equiv 1$ and $p = 2$:

$$-\Delta u + u = g(x,u) \qquad \text{in } \mathbb{R}^N$$

Variational setting

The natural action functional associated to problem (P) is

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N} A(x,u)|\nabla u|^p \, dx + \frac{1}{p}\int_{\mathbb{R}^N} |u|^p \, dx - \frac{1}{p}\int_{\mathbb{R}^N} G(x,u) \, dx$$

(here $G(x,t) = \int_0^t g(x,s) \, ds$)
which is not defined in $W^{1,p}(\mathbb{R}^N)$ for a general coefficient A in the principal part. We note that if $0 < \alpha_* \leq A(x,t) \leq \alpha_2$ but with $\frac{\partial A}{\partial t}(x,t) \neq 0$, functional J is well defined in $W^{1,p}(\mathbb{R}^N)$ but it is Gâteaux differentiable only along directions in the Banach space $X = W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.
Anyway, under suitable assumptions, J is C^1 in X equipped with

$$\|u\|_X = \|u\|_W + |u|_\infty$$

where $\|u\|_W = \left(|\nabla u|_p^p + |u|_p^p\right)^{\frac{1}{p}}$ and $|u|_\infty = \operatorname*{ess\,sup}_{\mathbb{R}^N} |u|$.

Variational setting

More precisely, assume that:

(H$_1$) $A(x,t)$ and $A_t(x,t)$ are essentially bounded if t is bounded, *i.e.*

$$\sup_{|t|\le r} |A(\cdot,t)| \in L^\infty(\mathbb{R}^N), \qquad \sup_{|t|\le r} |A_t(\cdot,t)| \in L^\infty(\mathbb{R}^N) \qquad \text{for any } r>0;$$

(G$_1$) $a_1, a_2 > 0$ and $q \ge p$ exist such that

$$|g(x,t)| \le a_1|t|^{p-1} + a_2|t|^{q-1} \qquad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \in \mathbb{R}.$$

The following regularity result holds:

Regularity result

Proposition (Regularity Result)

Assume that (H_1) and (G_1) hold. If $(u_n)_n \subset X$, $u \in X$, $M > 0$ are such that

- $\|u_n - u\|_W \to 0, \quad u_n \to u \quad$ a.e. in $\mathbb{R}^N$ if $n \to +\infty$,
- $|u_n|_\infty \le M \quad$ for all $n \in \mathbb{N}$,

then

$$J(u_n) \to J(u) \qquad \text{and} \qquad \|dJ(u_n) - dJ(u)\|_{X'} \to 0,$$

where for any $u, v \in X$ we have

$$\langle dJ(u), v\rangle = \int_{\mathbb{R}^N} A(x,u)|\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \frac{1}{p}\int_{\mathbb{R}^N} A_t(x,u) v |\nabla u|^p \, dx + \int_{\mathbb{R}^N} |u|^{p-2} uv \, dx - \int_{\mathbb{R}^N} g(x,u) v \, dx.$$

Remarks

Remark

From the previous proposition it follows that

- $J \in C^1(X, \mathbb{R})$
- Critical points of J in X are bounded weak solutions of problem (P).

Remark

Functional J does not verify the classical Palais-Smale condition in X as Palais-Smale sequences may exist which are unbounded in X but converge in the $W^{1,p}(\mathbb{R}^N)$-norm.
[**Candela - Palmieri**, *Calc. Var.* 2017]

wCPS condition: definition

Taking $\beta \in \mathbb{R}$, a sequence $(u_n)_n \subset X$ is a $(CPS)_\beta$-sequence if

$$\lim_n J(u_n) = \beta \qquad \text{and} \qquad \lim_n \|dJ(u_n)\|_{X'}(1 + \|u_n\|_X) = 0.$$

Definition

Functional J satisfies a weak version of the Cerami's variant of Palais-Smale condition at level β ($\beta \in \mathbb{R}$), briefly $(wCPS)_\beta$ condition, if taking any $(CPS)_\beta$-sequence $(u_n)_n$ a point $u \in X$ exists such that

(i) $\lim_n \|u_n - u\|_W = 0$ (up to subsequences),

(ii) $J(u) = \beta, \quad dJ(u) = 0$

wCPS condition: remarks

This weaker compactness condition is enough to prove a Deformation Lemma and then some abstract critical point theorems.

Theorem (generalized version of the Mountain Pass Theorem)

Let $J \in C^1(X, \mathbb{R})$ be such that $J(0) = 0$ and the (wCPS) condition holds in $\mathbb{R}$. Moreover, assume that two constants $r, \rho > 0$ and a point $e \in X$ exist such that

$$u \in X, \|u\|_W = r \quad \Longrightarrow \; J(u) \geq \rho$$

$$\|e\|_W > r \quad \text{and} \quad J(e) < \rho.$$

Then, J has a Mountain pass critical point $u^* \in X$ such that $J(u^*) \geq \rho$.
[**Candela**, **Palmieri** *Discrete and Continous Dynam. Systems, Suppl.*, 2009]

Main result

We consider the space

$$X_r = W^{1,p}_r(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$$

endowed with $\|\cdot\|_X$ and from now on, for simplicity, we still denote by J the restriction of J to X_r.

Theorem (Candela-Salvatore, Nonlinear Anal. 2020)

Assume that $A(x,t)$ and $g(x,t)$ satisfy (H$_1$), (G$_1$) and some positive constants α_0, α_1, α_2, R and $\mu > p$ exist s.t.

- (H$_2$) $A(x,t) \geq \alpha_0$ a.e. in $\mathbb{R}^N$, for all $t \in \mathbb{R}$,
- (H$_3$) $pA(x,t) + A_t(x,t)t \geq \alpha_1 A(x,t)$ a.e. in $\mathbb{R}^N$, for all $t \in \mathbb{R}$,
- (H$_4$) $(\mu - p)A(x,t) - A_t(x,t)t \geq \alpha_2 A(x,t)$ a.e. in $\mathbb{R}^N$, for all $t \in \mathbb{R}$,
- (H$_5$) $A(x,t) = A(|x|,t)$ a.e. in $\mathbb{R}^N$, for all $t \in \mathbb{R}$,
- (G$_2$) $\displaystyle\lim_{t\to 0} \frac{g(x,t)}{|t|^{p-1}} = 0$ uniformly for a.e. $x \in \mathbb{R}^N$,
- (G$_3$) $0 < \mu G(x,t) \leq g(x,t)$ a.e. in $\mathbb{R}^N$, for all $t \in \mathbb{R}$,
- (G$_4$) $g(x,t) = g(|x|,t)$ a.e. in $\mathbb{R}^N$, for all $t \in \mathbb{R}$.

Then, if $p < q < p^*$, (P) has at least one weak bounded nontrivial radial solution.

Main result

Remark

If we consider the particular coefficient

$$A(x,t) = A_1(x) + A_2(x)|t|^\gamma$$

with $A_1, A_2 \in L^\infty(\mathbb{R}^N)$, then the previous assumptions on A are verified if

$$\gamma > 1, \quad A_1(x) \geq \alpha_0, \quad A_2(x) \geq 0 \qquad \text{a.e. in } \mathbb{R}^N, \qquad A_1, A_2 \text{ radially symmetric.}$$

In particular, if $A(x,t) = 1$, we obtain the following result:

Corollary

Assume that $g(x,t)$ verify (G_1) – (G_4). Then, if $p < q < p^*$, the p-Laplacian equation

$$-\text{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{p-2}u = g(x,u) \quad \text{in } \mathbb{R}^N$$

has at least one weak bounded nontrivial radial solution.

Proof of the main result

By adapting the arguments in **Bérestycki-Lions**, *Arch. Ration. Mech. Anal.* 1983, the following results can be stated:

Proposition (Radial Lemma)

If $p > 1$, then every radial function $u \in W_r^{1,p}(\mathbb{R}^N)$ is almost everywhere equal to a function $U(x)$, continous for $x \neq 0$, s.t.

$$|U(x)| \leq C\frac{\|u\|_W}{|x|^\theta} \qquad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geq 1$$

for suitable constants $C, \theta > 0$ depending only on N and p.

Proposition (Compact Imbedding)

If $p > 1$, then the following compact impeddings hold:

$$W_r^{1,p}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^\ell(\mathbb{R}^N) \qquad \text{for any } p < \ell < p^*.$$

Proof of the main Theorem

In order to apply the Generalized Version of the Mountain Pass Theorem, we need to prove that J verifies the (wCPS) condition in $\mathbb{R}$ if $p < q < p^*$.

Let $(u_n)_n$ be a $(CPS)_\beta$-sequence ($\beta \in \mathbb{R}$). It is easy to prove that $(u_n)_n$ is bounded in $W^{1,p}_r(\mathbb{R}^N)$ then, from the Radial Lemma, an uniform estimate holds, *i.e.*

$$|u_n(x)| \leq \beta_0 \quad \text{for a.e. } x \in \mathbb{R}^N, \qquad |x| \geq 1, \quad \text{for all } n \in \mathbb{N}.$$

Moreover, there exist $u \in W^{1,p}_r(\mathbb{R}^N)$ s.t., up to subsequences,

$$\begin{aligned} u_n &\rightharpoonup u && \text{weakly in } W^{1,p}_r(\mathbb{R}^N) \\ u_n &\to u && \text{strongly in } L^\ell(\mathbb{R}^N) \text{ for each } \ell \in \,]p, p^*[\\ u_n &\to u && \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Clearly, the Radial Lemma implies that

$$\operatorname*{ess\,sup}_{|x|\geq 1} |u(x)| \quad \text{is finite.}$$

Existence of solutions

Furthermore, adapting the arguments developed by **Candela-Palmieri** (*Advanced Nonlinear Stud.* 2006, *Calculus of Variations* 2009) we prove that

- u is bounded on the bounded sets, hence $u \in L^\infty(\mathbb{R}^N)$
- $\|u_n - u\|_W \to 0 \quad \text{if } n \to \infty$
- $J(u) = \beta \quad \text{and} \quad \mathrm{d}J(u) = 0$,

i.e. $(\text{wCPS})_\beta$ holds.

Finally, we note that J has the Mountain Pass geometry, then a mountain pass critical point of J in X_r exists and the existence of at least one nontrivial radial bounded solution follows.

Papers in preparation

- **A. M. Candela, G. Palmieri, A. Salvatore**
  Existence of solutions of Modified Schrödinger equations on unbounded domains
- **A. M. Candela, A. Salvatore, C. Sportelli**
  Existence and multiplicity results for some weighted quasilinear elliptic equation in $\mathbb{R}^N$

Thank you for your attention