Local asymptotics for some q-hypergeometric polynomials

Author: Juan Francisco Mañas Mañas

UNIVERSIDAD DE ALMERÍA

Joint work with Juan J. Moreno-Balcázar (UAL)

Introduction

We consider the q-hypergeometric function ${}_r\phi_s$ defined by the series

$$
{}_r\phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,z\right)
= \sum_{k=0}^{\infty} \frac{(a_1;q)_k\,(a_2;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\,(b_2;q)_k\cdots(b_s;q)_k}(-1)^{(1+s-r)k}q^{(1+s-r)\binom{k}{2}}\frac{z^k}{(q;q)_k},
$$

with $0<q<1$ and the expressions $(a_j;q)_k$ and $(b_j;q)_k$ denote the q-analogues of the Pochhammer symbol, i.e.

$$
(a;q)_0 := 1 \text{ and } (a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right), \quad n\geq 1,
$$

$$
(a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right).
$$

We are going to establish the following asymptotic behavior

$$\lim_{n\to+\infty} {}_s\phi_s \left(\begin{array}{c} q^{-n}, q^{a_1 n+b_1}, q^{a_2 n+b_2}, \ldots, q^{a_{s-1} n+b_{s-1}} \\ q^{\alpha}, q^{c_1 n+d_1}, q^{c_2 n+d_2}, \ldots, q^{c_{s-1} n+d_{s-1}} \end{array} ; q, \frac{z(q-1)}{q^{-n-\alpha}[n]_q \frac{[n]_{q^{a_1}}[n]_{q^{a_2}}\cdots[n]_{q^{a_{s-1}}}}{[n]_{q^{c_1}}[n]_{q^{c_2}}\cdots[n]_{q^{c_{s-1}}}}} \right)$$

$$= 2^{\alpha-1}\Gamma_q(\alpha) \left(2\sqrt{\frac{z[a_1]_q[a_2]_q\cdots[a_{s-1}]_q}{[c_1]_q[c_2]_q\cdots[c_{s-1}]_q}} \right)^{1-\alpha} J^{(2)}_{\alpha-1}\left(2\sqrt{\frac{z[a_1]_q[a_2]_q\cdots[a_{s-1}]_q}{[c_1]_q[c_2]_q\cdots[c_{s-1}]_q}(1-q)}; q \right),$$

where

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \qquad [z]_q = \frac{1-q^z}{1-q}, \qquad 0<q<1,$$

and

$$J^{(2)}_\alpha(z;q) = \frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}\left(\frac{z}{2}\right)^\alpha {}_0\phi_1\left(\begin{array}{c} - \\ q^{\alpha+1} \end{array} ; q, \frac{-q^{\alpha+1}z^2}{4} \right).$$

Technical Results

Lemma 1^{st}

Let k be a nonnegative integer and z a complex number, then

$$\frac{\Gamma_q(z+k)}{\Gamma_q(z)} = \frac{(q^z;q)_k}{(1-q)^k}.$$

Technical Results

Lemma 1st

Let k be a nonnegative integer and z a complex number, then

$$\frac{\Gamma_q(z+k)}{\Gamma_q(z)} = \frac{(q^z;q)_k}{(1-q)^k}.$$

Lemma 2nd

Let a be a positive real number and b a complex number. Then, we have, for any $k \in \mathbb{Z}$ fixed,

$$\lim_{n\to+\infty} \frac{\Gamma_q(an+b+k)}{\Gamma_q(an+b)} [an+b]_q^{-k} = 1.$$

Technical Results

Lemma 1^{st}

Let k be a nonnegative integer and z a complex number, then

$$\frac{\Gamma_q(z+k)}{\Gamma_q(z)} = \frac{(q^z;q)_k}{(1-q)^k}.$$

Lemma 2^{nd}

Let a be a positive real number and b a complex number. Then, we have, for any $k \in \mathbb{Z}$ fixed,

$$\lim_{n\to+\infty} \frac{\Gamma_q(an+b+k)}{\Gamma_q(an+b)} [an+b]_q^{-k} = 1.$$

We use,

$$\Gamma_q(z+1) = [z]_q \Gamma_q(z) \qquad \text{and} \qquad \lim_{n\to+\infty} \frac{[an+b+k]_q}{[an+b]_q} = 1.$$

Lemma 3rd

Let k be a nonnegative integer. Then,

$$\lim_{n\to+\infty}\frac{(q^{-n};q)_k}{[n]_q^k q^{-nk}}=(-1)^k q^{\binom{k}{2}}(1-q)^k.$$

Technical Results

Lemma 3rd

Let k be a nonnegative integer. Then,

$$\lim_{n\to+\infty} \frac{(q^{-n};q)_k}{[n]_q^k q^{-nk}} = (-1)^k q^{\binom{k}{2}} (1-q)^k.$$

Lemma 4th

Let b be a complex number. Then, for a positive real number a, it holds

$$\lim_{n\to+\infty} \frac{\left(q^{an+b};q\right)_k}{(1-q)^k [n]_{q^a}^k} = [a]_q^k.$$

Technical Results

Proposition

We have for $n \geq 1$,

$$\left| \frac{(q^{-n};q)_k}{[n]_q^k q^{-nk}} \right| \leq q^{\binom{k}{2}}, \quad k = 0, 1, \ldots, n.$$

Proposition

Let a *be a positive real number and* $b = \gamma + i\beta$ *a complex number. We assume that* $an + \gamma \notin \mathbb{Z}_*$ *for all* n *positive integer where* $\mathbb{Z}_* = \{0, -1, -2, \ldots\}$. *Then, for* $n \geq 1$ *and* $k = 0, 1, \ldots, n$, *we can affirm that there are two constants,* $\mathfrak{C}_a$ *and* $\mathfrak{D}_a$, *independent of* n, *so that*

$$0 < \mathfrak{C}_a^k \leq \left| \frac{(q^{an+b};q)_k}{(1-q)^k [n]_{q^a}^k} \right| \leq \mathfrak{D}_a^k.$$

Main Result

Proposition

Let $\alpha \in \mathbb{R}\backslash\mathbb{Z}_*$ *be. Then,*

$$2^{\alpha-1}\Gamma_q(\alpha)(2\sqrt{z})^{1-\alpha}J_{\alpha-1}^{(2)}(2\sqrt{z}(1-q);q) = {}_0\phi_1\left(\begin{matrix} - \\ q^\alpha \end{matrix};q,-z(q-1)^2q^\alpha\right).$$

Main Result

Proposition

Let $\alpha \in \mathbb{R}\backslash\mathbb{Z}_*$ *be. Then,*

$$2^{\alpha-1}\Gamma_q(\alpha)(2\sqrt{z})^{1-\alpha}J^{(2)}_{\alpha-1}(2\sqrt{z}(1-q);q) = {}_0\phi_1\left(\begin{array}{c} - \\ q^{\alpha}\end{array};q,-z(q-1)^2q^{\alpha}\right).$$

Theorem (Mehler–Heine asymptotics)

Assuming that $\alpha \in \mathbb{R}\backslash\mathbb{Z}_*$, $a_j > 0$, $c_j > 0$ and that b_j and d_j are complex numbers satisfying $a_j n + \mathrm{Re}(b_j) \notin \mathbb{Z}_*$ and $c_j n + \mathrm{Re}(d_j) \notin \mathbb{Z}_*$ with $j \in \{1,2,\ldots,s-1\}$, then

$$\lim_{n\to+\infty} {}_s\phi_s\left(\begin{array}{c} q^{-n}, q^{a_1 n+b_1}, q^{a_2 n+b_2}, \cdots, q^{a_{s-1}n+b_{s-1}} \\ q^{\alpha}, q^{c_1 n+d_1}, q^{c_2 n+d_2}, \cdots, q^{c_{s-1}n+d_{s-1}}\end{array};q,\frac{z(q-1)}{q^{-n-\alpha}[n]_q\frac{[n]_{q^{a_1}}[n]_{q^{a_2}}\cdots[n]_{q^{a_{s-1}}}}{[n]_{q^{c_1}}[n]_{q^{c_2}}\cdots[n]_{q^{c_{s-1}}}}}\right)$$

$$= 2^{\alpha-1}\Gamma_q(\alpha)\left(2\sqrt{\frac{z[a_1]_q[a_2]_q\cdots[a_{s-1}]_q}{[c_1]_q[c_2]_q\cdots[c_{s-1}]_q}}\right)^{1-\alpha} J^{(2)}_{\alpha-1}\left(2\sqrt{\frac{z[a_1]_q[a_2]_q\cdots[a_{s-1}]_q}{[c_1]_q[c_2]_q\cdots[c_{s-1}]_q}}(1-q);q\right)$$

uniformly on compact subsets of the complex.

Main Result

Sketch of the proof

1. Scaling the variable z, then

$$
{}_s\phi_s\left(\begin{matrix} q^{-n}, q^{a_1 n+b_1}, q^{a_2 n+b_2}, \dots, q^{a_{s-1} n+b_{s-1}} \\ q^{\alpha}, q^{c_1 n+d_1}, q^{c_2 n+d_2}, \dots, q^{c_{s-1} n+d_{s-1}} \end{matrix}; q, \frac{z(q-1)}{q^{-n-\alpha}[n]_q \frac{[n]_{q^{a_1}}[n]_{q^{a_2}}\cdots[n]_{q^{a_{s-1}}}}{[n]_{q^{c_1}}[n]_{q^{c_2}}\cdots[n]_{q^{c_{s-1}}}}}\right)
$$

$$
= \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{a_1 n+b_1};q)_k \cdots (q^{a_{s-1} n+b_{s-1}};q)_k}{(q^{\alpha};q)_k (q^{c_1 n+d_1};q)_k \cdots (q^{c_{s-1} n+d_{s-1}};q)_k} (-1)^k q^{\binom{k}{2}} \frac{z^k q^{\alpha k}(q-1)^k}{q^{-nk}[n]_q^k \frac{[n]^k_{q^{a_1}}[n]^k_{q^{a_2}}\cdots[n]^k_{q^{a_{s-1}}}}{[n]^k_{q^{c_1}}[n]^k_{q^{c_2}}\cdots[n]^k_{q^{c_{s-1}}}}(q;q)_k}
$$

$$
:= \sum_{k=0}^{n} g_{n,k}(z).
$$

Main Result

Sketch of the proof

1. Scaling the variable z, then

$$
{}_s\phi_s\left(\begin{array}{c} q^{-n}, q^{a_1 n+b_1}, q^{a_2 n+b_2}, \dots, q^{a_{s-1} n+b_{s-1}} \\ q^{\alpha}, q^{c_1 n+d_1}, q^{c_2 n+d_2}, \dots, q^{c_{s-1} n+d_{s-1}} \end{array} ; q, \frac{z(q-1)}{q^{-n-\alpha}[n]_q \frac{[n]_{q^{a_1}}[n]_{q^{a_2}}\cdots[n]_{q^{a_{s-1}}}}{[n]_{q^{c_1}}[n]_{q^{c_2}}\cdots[n]_{q^{c_{s-1}}}}}\right)
$$

$$
= \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{a_1 n+b_1};q)_k \cdots (q^{a_{s-1} n+b_{s-1}};q)_k}{(q^{\alpha};q)_k (q^{c_1 n+d_1};q)_k \cdots (q^{c_{s-1} n+d_{s-1}};q)_k} (-1)^k q^{\binom{k}{2}} \frac{z^k q^{\alpha k}(q-1)^k}{q^{-nk}[n]_q^k \frac{[n]_{q^{a_1}}^k [n]_{q^{a_2}}^k \cdots [n]_{q^{a_{s-1}}}^k}{[n]_{q^{c_1}}^k [n]_{q^{c_2}}^k \cdots [n]_{q^{c_{s-1}}}^k}(q;q)_k}
$$

$$
:= \sum_{k=0}^{n} g_{n,k}(z).
$$

2. We have for k fixed,

$$
\lim_{n\to+\infty} g_{n,k}(z) = q^{2\binom{k}{2}}(1-q)^k \frac{[a_1]_q^k [a_2]_q^k \cdots [a_{s-1}]_q^k}{[c_1]_q^k [c_2]_q^k \cdots [c_{s-1}]_q^k} \frac{z^k q^{\alpha k}(-1)^k (1-q)^k}{(q^{\alpha};q)_k (q;q)_k}.
$$

Sketch of the proof

3. Taking z on a compact subset of the complex plane, then $|z| \leq \mathfrak{C}_\Omega$ and we obtain $\left|g_{n,k}(z)\right| \leq \frac{\mathfrak{D}_{a_1}^k \mathfrak{D}_{a_2}^k \cdots \mathfrak{D}_{a_{s-1}}^k}{\mathfrak{C}_{c_1}^k \mathfrak{C}_{c_2}^k \cdots \mathfrak{C}_{c_{s-1}}^k} \frac{q^{2\binom{k}{2}} q^{\alpha k}(1-q)^k}{\left|(q^\alpha;q)_k\right|(q;q)_k} \mathfrak{C}_\Omega^k$.

Sketch of the proof

3. Taking z on a compact subset of the complex plane, then $|z| \leq \mathfrak{C}_\Omega$ and we obtain $|g_{n,k}(z)| \leq \frac{\mathfrak{D}_{a_1}^k \mathfrak{D}_{a_2}^k \cdots \mathfrak{D}_{a_{s-1}}^k}{\mathfrak{C}_{c_1}^k \mathfrak{C}_{c_2}^k \cdots \mathfrak{C}_{c_{s-1}}^k} \frac{q^{2\binom{k}{2}} q^{\alpha k}(1-q)^k}{|(q^\alpha;q)_k|(q;q)_k} \mathfrak{C}_\Omega^k$. Thus, the previous series is dominated by $\sum_{k=0}^{+\infty} \frac{\mathfrak{D}_{a_1}^k \mathfrak{D}_{a_2}^k \cdots \mathfrak{D}_{a_{s-1}}^k}{\mathfrak{C}_{c_1}^k \mathfrak{C}_{c_2}^k \cdots \mathfrak{C}_{c_{s-1}}^k} \frac{q^{2\binom{k}{2}} q^{\alpha k}(1-q)^k}{|(q^\alpha;q)_k|(q;q)_k} \mathfrak{C}_\Omega^k$, and this series is convergent.

Sketch of the proof

3. Taking z on a compact subset of the complex plane, then $|z| \leq \mathfrak{C}_\Omega$ and we obtain $|g_{n,k}(z)| \leq \frac{\mathfrak{D}_{a_1}^k \mathfrak{D}_{a_2}^k \cdots \mathfrak{D}_{a_{s-1}}^k}{\mathfrak{C}_{c_1}^k \mathfrak{C}_{c_2}^k \cdots \mathfrak{C}_{c_{s-1}}^k} \frac{q^{2\binom{k}{2}} q^{\alpha k}(1-q)^k}{|(q^\alpha;q)_k|(q;q)_k} \mathfrak{C}_\Omega^k$. Thus, the previous series is dominated by $\sum_{k=0}^{+\infty} \frac{\mathfrak{D}_{a_1}^k \mathfrak{D}_{a_2}^k \cdots \mathfrak{D}_{a_{s-1}}^k}{\mathfrak{C}_{c_1}^k \mathfrak{C}_{c_2}^k \cdots \mathfrak{C}_{c_{s-1}}^k} \frac{q^{2\binom{k}{2}} q^{\alpha k}(1-q)^k}{|(q^\alpha;q)_k|(q;q)_k} \mathfrak{C}_\Omega^k$, and this series is convergent.
4. We can apply the Lebesgue's dominated convergence theorem to obtain

$$\lim_{n\to+\infty} \sum_{k=0}^{n} \frac{\left(q^{-n};q\right)_k \left(q^{a_1 n+b_1};q\right)_k \cdots \left(q^{a_{s-1} n+b_{s-1}};q\right)_k}{\left(q^{\alpha};q\right)_k \left(q^{c_1 n+d_1};q\right)_k \cdots \left(q^{c_{s-1} n+d_{s-1}};q\right)_k} \frac{(-1)^k q^{\binom{k}{2}} z^k q^{\alpha k}(q-1)^k}{q^{-nk}[n]_q^k \frac{[n]_{q^{a_1}}^k [n]_{q^{a_2}}^k \cdots [n]_{q^{a_{s-1}}}^k}{[n]_{q^{c_1}}^k [n]_{q^{c_2}}^k \cdots [n]_{q^{c_{s-1}}}^k} (q;q)_k}$$

$$= \lim_{n\to+\infty} \sum_{k=0}^{n} g_{n,k}(z) = \sum_{k=0}^{+\infty} (-1)^k q^{2\binom{k}{2}} \frac{[a_1]_q^k [a_2]_q^k \cdots [a_{s-1}]_q^k}{[c_1]_q^k [c_2]_q^k \cdots [c_{s-1}]_q^k} \frac{z^k q^{\alpha k}(1-q)^{2k}}{(q^\alpha;q)_k (q;q)_k}$$

Sketch of the proof

3. Taking z on a compact subset of the complex plane, then $|z| \leq \mathfrak{C}_\Omega$ and we obtain $|g_{n,k}(z)| \leq \frac{\mathfrak{D}_{a_1}^k \mathfrak{D}_{a_2}^k \cdots \mathfrak{D}_{a_{s-1}}^k}{\mathfrak{C}_{c_1}^k \mathfrak{C}_{c_2}^k \cdots \mathfrak{C}_{c_{s-1}}^k} \frac{q^{2\binom{k}{2}} q^{\alpha k}(1-q)^k}{|(q^\alpha;q)_k|(q;q)_k} \mathfrak{C}_\Omega^k$. Thus, the previous series is dominated by $\sum_{k=0}^{+\infty} \frac{\mathfrak{D}_{a_1}^k \mathfrak{D}_{a_2}^k \cdots \mathfrak{D}_{a_{s-1}}^k}{\mathfrak{C}_{c_1}^k \mathfrak{C}_{c_2}^k \cdots \mathfrak{C}_{c_{s-1}}^k} \frac{q^{2\binom{k}{2}} q^{\alpha k}(1-q)^k}{|(q^\alpha;q)_k|(q;q)_k} \mathfrak{C}_\Omega^k$, and this series is convergent.

4. We can apply the Lebesgue's dominated convergence theorem to obtain

$$\lim_{n\to+\infty} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{a_1 n+b_1};q)_k \cdots (q^{a_{s-1} n+b_{s-1}};q)_k}{(q^\alpha;q)_k (q^{c_1 n+d_1};q)_k \cdots (q^{c_{s-1} n+d_{s-1}};q)_k} \frac{(-1)^k q^{\binom{k}{2}} z^k q^{\alpha k}(q-1)^k}{q^{-nk}[n]_q^k \frac{[n]_{q^{a_1}}^k [n]_{q^{a_2}}^k \cdots [n]_{q^{a_{s-1}}}^k}{[n]_{q^{c_1}}^k [n]_{q^{c_2}}^k \cdots [n]_{q^{c_{s-1}}}^k} (q;q)_k}$$

$$= \lim_{n\to+\infty} \sum_{k=0}^{n} g_{n,k}(z) = \sum_{k=0}^{+\infty} (-1)^k q^{2\binom{k}{2}} \frac{[a_1]_q^k [a_2]_q^k \cdots [a_{s-1}]_q^k}{[c_1]_q^k [c_2]_q^k \cdots [c_{s-1}]_q^k} \frac{z^k q^{\alpha k}(1-q)^{2k}}{(q^\alpha;q)_k (q;q)_k}$$

$$= {}_0\phi_1\left(\begin{array}{c} - \\ q^\alpha \end{array} ; q, -\frac{z[a_1]_q[a_2]_q \cdots [a_{s-1}]_q}{[c_1]_q[c_2]_q \cdots [c_{s-1}]_q} (q-1)^2 q^\alpha \right).$$

Proposition

We take $\mathbf{r} < \mathbf{s}$ *and* $\alpha \in \mathbb{R}\backslash\mathbb{Z}_*$. *We consider* b_j *and* d_ℓ *complex numbers satisfying* $a_j n + \mathrm{Re}(b_j) \notin \mathbb{Z}_*$, $c_\ell n + \mathrm{Re}(d_\ell) \notin \mathbb{Z}_*$ *where* $a_j > 0$, $c_\ell > 0$ *with* $j \in \{1, 2, \ldots, r-1\}$ *and* $\ell \in \{1, 2, \ldots, s-1\}$. *Then, we have*

$$\lim_{n\to+\infty} {}_r\phi_s\left(\begin{matrix} q^{-n}, q^{a_1 n+b_1}, q^{a_2 n+b_2}, \cdots, q^{a_{r-1} n+b_{r-1}} \\ q^{\alpha}, q^{c_1 n+d_1}, q^{c_2 n+d_2}, \cdots, q^{c_{s-1} n+d_{s-1}} \end{matrix}; q, \frac{z(q-1)}{q^{-n-\alpha}[n]_q \frac{[n]_{q^{a_1}}[n]_{q^{a_2}}\cdots[n]_{q^{a_{r-1}}}}{[n]_{q^{c_1}}[n]_{q^{c_2}}\cdots[n]_{q^{c_{s-1}}}}}\right)$$

$$= \sum_{k=0}^{+\infty} (-1)^{(1+r-s)k} q^{(2+s-r)\binom{k}{2}} \frac{[a_1]_q^k[a_2]_q^k\cdots[a_{r-1}]_q^k}{[c_1]_q^k[c_2]_q^k\cdots[c_{s-1}]_q^k} \frac{z^k q^{\alpha k}(1-q)^{(2+r-s)k}}{(q^\alpha;q)_k\,(q;q)_k}.$$

Remarks

Remark 1

In (Bracciali–Moreno–Balcázar, AMC, 2015) the authors obtain a Melher-Heine formula for some hypergeometric polynomials. Taking into account

$$\lim_{q\to 1} {}_r\phi_s\left(\begin{array}{c} q^{a_1},\dots,q^{a_r} \\ q^{b_1},\dots,q^{b_s} \end{array}; q, (q-1)^{1+s-r} z\right) = {}_rF_s\left(\begin{array}{c} a_1,\dots,a_r \\ b_1,\dots,b_s \end{array}; z\right).$$

our theorem recovers partially their results.

Remarks

Remark 1

In (Bracciali–Moreno–Balcázar, AMC, 2015) the authors obtain a Melher-Heine formula for some hypergeometric polynomials. Taking into account

$$\lim_{q\to 1} {}_r\phi_s\left(\begin{array}{c} q^{a_1},\dots,q^{a_r} \\ q^{b_1},\dots,q^{b_s}\end{array};q,(q-1)^{1+s-r}z\right) = {}_rF_s\left(\begin{array}{c} a_1,\dots,a_r \\ b_1,\dots,b_s\end{array};z\right).$$

our theorem recovers partially their results.

Remark 2

Using our theorem we can obtain a well-known type of asymptotics for the q-Laguerre orthogonal polynomials, given by

$$\lim_{n\to+\infty} L_n^{(\alpha)}(z;q) = z^{-\alpha/2} J_\alpha^{(2)}\left(2\sqrt{z};q\right),$$

uniformly on compact subsets of the complex plane.

Consequence about the zeros

We denote by $x_{q,n,k}$ with $1 \leq k \leq n$ the zeros of the function

$$ {}_s\phi_s \left(\begin{array}{c} q^{-n}, q^{a_1 n+b_1}, q^{a_2 n+b_2}, \dots, q^{a_{s-1} n+b_{s-1}} \\ q^{\alpha}, q^{c_1 n+d_1}, q^{c_2 n+d_2}, \dots, q^{c_{s-1} n+d_{s-1}} \end{array} ; q, z(q-1) \right), $$

and by z_ℓ the zeros of the limit function

$$ \left(2\sqrt{\frac{z[a_1]_q[a_2]_q \cdots [a_{s-1}]_q}{[c_1]_q[c_2]_q \cdots [c_{s-1}]_q}} \right)^{1-\alpha} J_{\alpha-1}^{(2)} \left(2\sqrt{\frac{z[a_1]_q[a_2]_q \cdots [a_{s-1}]_q}{[c_1]_q[c_2]_q \cdots [c_{s-1}]_q}(1-q); q \right). $$

Then, using the Hurwitz's Theorem, we have that the scaled zeros of the function ${}_s\phi_s$

$$ x^*_{q,n,k} := q^{-n-\alpha}[n]_q \frac{[n]_{q^{a_1}}[n]_{q^{a_2}} \cdots [n]_{q^{a_{s-1}}}}{[n]_{q^{c_1}}[n]_{q^{c_2}} \cdots [n]_{q^{c_{s-1}}}} x_{q,n,k}, $$

converge to the zeros of the limit function.

We take the following data: $s = 3$, $q = 1/2$, $\alpha = 1$ and

$a_1 = 3$	$b_1 = 6$	$c_1 = 4/3$	$d_1 = 2 - 3i$
$a_2 = 5/4$	$b_2 = -2/3 + 2i$	$c_2 = 5/6$	$d_2 = 1$

	$x^*_{1/2,n,1}$
$n = 10$	$1.192838457108567 - 0.000373946558540i$
$n = 20$	$1.191325673237055 - 6.242164358332549 \times 10^{-8} i$
$n = 40$	$1.191320585493521 - 1.806507624485436 \times 10^{-15} i$
z_ℓ	1.191320585443003

	$x^*_{1/2,n,2}$
$n = 10$	$10.5677725749753241 - 0.001544756947360i$
$n = 20$	$10.5384014795907282 - 2.574926397461865 \times 10^{-7} i$
$n = 40$	$10.5383205870081968 - 7.451913839285253 \times 10^{-15} i$
z_ℓ	10.5383205862744722

	$x^*_{1/2,n,3}$
$n = 10$	$54.673251810109320 - 0.003501823799542i$
$n = 20$	$54.420541178838059 - 5.826491777755416 \times 10^{-7} i$
$n = 40$	$54.419973744203762 - 1.686199155483337 \times 10^{-15} i$
z_ℓ	54.419973739631614

We take the following data: $s = 3$, $q = 1/2$, $\alpha = -51/100$ and

$a_1 = 3$	$b_1 = 6$	$c_1 = 4/3$	$d_1 = 2 - 3i$
$a_2 = 5/4$	$b_2 = -2/3 + 2i$	$c_2 = 5/6$	$d_2 = 1$

	$x^*_{1/2,n,1}$
$n = 10$	$-0.257907814868696 + 0.000069310548527i$
$n = 20$	$-0.257444571262302 + 1.156063483173270 \times 10^{-8}i$
$n = 40$	$-0.257443205893177 + 3.345689303361298 \times 10^{-16}i$
z_ℓ	-0.257443205880353

	$x^*_{1/2,n,2}$
$n = 10$	$1.717581648483185 - 0.000304499739637i$
$n = 20$	$1.712978988207565 - 5.076635738362011 \times 10^{-8}i$
$n = 40$	$1.712966620706365 - 1.469194883628405 \times 10^{-15}i$
z_ℓ	1.712966620595486

	$x^*_{1/2,n,3}$
$n = 10$	$15.162942911385623 - 0.001127913055563i$
$n = 20$	$15.093385289370837 - 1.877379036189081 \times 10^{-7}i$
$n = 40$	$15.093230236139474 - 5.433179181138827 \times 10^{-15}i$
z_ℓ	15.093230234896025

We take the following data: $s = 3$, $q = 1/2$, $\alpha = -78/10$ and

$a_1 = 3$	$b_1 = 6$	$c_1 = 4/3$	$d_1 = 2 - 3i$
$a_2 = 5/4$	$b_2 = -2/3 + 2i$	$c_2 = 5/6$	$d_2 = 1$

	$x^*_{1/2,n,1}$
$n = 10$	$-26.583782080249094 + 13.144830793624180i$
$n = 20$	$-24.217517881848614 + 13.495110134590402i$
$n = 40$	$-24.215268341966693 + 13.495244741512639i$
z_ℓ	$-24.215268337780208 + 13.495244740537485i$

	$x^*_{1/2,n,3}$
$n = 10$	$-11.012428251874263 + 62.145264518015616i$
$n = 20$	$-6.977745534060112 + 56.644723043027267i$
$n = 40$	$-6.974490887747343 + 56.639531189402158i$
z_ℓ	$-6.974490884027720 + 56.639531179755917i$

	$x^*_{1/2,n,5}$
$n = 10$	$-16.166116185151479 - 0.00004242253437i$
$n = 20$	$-15.859752369735673 - 9.93118142529022 \times 10^{-9} i$
$n = 40$	$-15.859327530746363 - 2.87493401650658 \times 10^{-16} i$
z_ℓ	-15.859327529012374

Consequence about the zeros

Numerical Experiments

	$x^*_{1/2,n,6}$
$n=10$	$-8.372365289781465 + 1.061870351231816 \times 10^{-6} i$
$n=20$	$-8.358589490583791 + 2.877523388911743 \times 10^{-10} i$
$n=40$	$-8.358521100270454 + 8.331309933918279 \times 10^{-18} i$
z_ℓ	-8.358521099509627

	$x^*_{1/2,n,7}$
$n=10$	$-4.183493772518376 + 7.184651279381444 \times 10^{-6} i$
$n=20$	$-4.174952644245207 + 1.169874676784201 \times 10^{-12} i$
$n=40$	$-4.174916567642287 - 8.016528868055206 \times 10^{-20} i$
z_ℓ	-4.174916567260315

	$x^*_{1/2,n,8}$
$n=10$	$-2.085524807895160 + 0.001212554068904 i$
$n=20$	$-2.087470339682883 + 2.036177391751408 \times 10^{-7} i$
$n=40$	$-2.087472421385608 + 5.892871393618265 \times 10^{-15} i$
z_ℓ	-2.087472421388369

References

- C. F. Bracciali, J. J. Moreno–Balcázar, *On the zeros of a class of generalized hypergeometric polynomials*, Appl. Math. Comput. **253** (2015), 151–158.
- M. E. H. Ismail, *The Zeros of Basic Bessel Functions, the Functions $J_{v+ax}(x)$, and Associated Orthogonal Polynomials*, J. Math. Anal. Appl. **86** (1982), 1–19.
- M. E. H. Ismail, Classical and Quantum Orthogonal Polynomials in one Variable. Enciclopedia of Mathematics and its Applications Vol. **98** Cambridge University Press, Cambridge U.K. 2005.
- R. Koekoek, P. A. Lesky, R. F. Swarttouw, Hypergeometric Orthogonal Polynomials and Their q–Analogues, Springer Monographs in Mathematics, Springer, Berlin, 2010.
- J. F. Mañas–Mañas, J. J. Moreno–Balcázar, *Asymptotics for some q-hypergeometric polynomials*, submmitted.
- D. S. Moak, *The q–Analogue of the Laguerre Polynomials*, J. Math. Anal. Appl. **81** (1981), 20–47.

THANKS FOR YOUR ATTENTION