On primes, almost primes and the Möbius function in short intervals

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June 22nd, 2021

Contents

What are primes?

- Write $\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, \dots\}$ for the set of primes, i.e. natural numbers > 1 that are only divisible by 1 and themselves.
- In this talk, p with or without subscripts is always a prime.

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- Euclid (c. 300 BC) showed that there are infinitely many primes: Suppose only $p_1, \ldots, p_k$ were primes. Then $p_1 \cdots p_k + 1$ is either a new prime or divisible by a new prime. Hence we obtain a contradiction.

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- Every integer can be uniquely written as a product of primes, e.g. $2021 = 43 \cdot 47$ — primes are the building blocks of the integers.

But are primes really like building blocks?

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- You need more and more primes to construct all integers
- Same for Legos — the kids want more and more!

How many primes are there?

- Hadamard and de la Vallee Poussin showed independently in 1896 that the number of primes up to x is

$$(1+o(1))\int_2^x \frac{dx}{\log x} = (1+o(1))\frac{x}{\log x}.$$

- This is called the prime number theorem (PNT).
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- It asserts that the "probability" that an integer n is prime is about $1/\log n$.
- Hence it is convenient to normalize prime p by $\log p$. More precisely we write $\Lambda(n)$ for the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } k \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

- Now PNT is equivalent to $\sum_{n \leq x} \Lambda(n) = (1+o(1))x$.

Relation to zeta zeros

- Write, for $\Re s > 1$,

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

This can be analytically continued to the whole complex plane except for a pole at $s = 1$. $\zeta(s)$ is called the Riemann zeta function.

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$$\text{PNT} \iff \sum_{n \leq x} \Lambda(n) = (1 + o(1))x \iff \zeta(s) \neq 0 \text{ when } \Re s = 1$$

$$\text{RH} \iff \sum_{n \leq x} \Lambda(n) = x + O(x^{1/2+\varepsilon}) \text{ for all } \varepsilon > 0.$$

Why zeta zeros are related to primes?

One has

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\zeta(s) = \frac{d}{ds}\log\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^s}\right) = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$$

Furthermore by contour integration

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{y^s}{s}ds = \begin{cases}1 & \text{if } y<1;\\ 0 & \text{if } y>1.\end{cases}$$

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These give (when $x \notin \mathbb{N}$)

$$\sum_{n\leq x}\Lambda(n) = \sum_{n}\Lambda(n)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{(x/n)^s}{s}ds = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds.$$

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- Moving the integration to the left one picks up a pole at $s = 1$ corresponding to the main term x and further poles at zeros of zeta that affect error terms.

This way one obtains

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Infinitude of primes from zeta

- One can see the infinitude of the primes also from

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  for $\Re s > 1$: Taking logs and letting $s \to 1$ one actually sees that $\sum_{p \in \mathbb{P}} \frac{1}{p}$ diverges.
- Analogously, if the number of buildings keeps tending to infinity, so must the number of bricks.

Primes in short intervals

- One wants to know about primes in short intervals. Huxley's prime number theorem from 1972 gives

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- This is based on Huxley's zero-density estimate

$$N(\sigma, T) = O\left(T^{(\frac{12}{5}+\varepsilon)(1-\sigma)}\right) \quad \text{for all } T \geq 2 \text{ and } \sigma \in [1/2, 1],$$

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  where $N(\sigma, T)$ is the number of zeros of the Riemann zeta function in the rectangle $\Re(s) \geq \sigma$, $|\Im(s)| \leq T$.
- This has resisted improvements, except Heath-Brown (1988) has shown $H \geq x^{7/12-o(1)}$.

Primes in short intervals

- Baker-Harman-Pintz (2001) showed with a sieve method

$$\sum_{x<n\leq x+H} \Lambda(n) \geq \varepsilon H, \quad H \geq x^{0.525}$$

  for some $\varepsilon > 0$.
- For shorter intervals one does not even know existence of primes!
- Assuming RH one knows that $[x, x + x^{1/2} \log x]$ always contains primes.

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- For shorter intervals one does not even know existence of primes!
- Assuming RH one knows that $[x, x + x^{1/2} \log x]$ always contains primes.
- Cramer made a probabilistic model based on "probability of n being prime is $1/\log n$". Based on this, one expects that intervals $[x, x + (\log x)^{2+\varepsilon}]$ contain primes for all large x.
- Huge gap between what's known and what's expected.

Primes in almost all short intervals

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- A lower bound has been shown for $H \geq X^{1/20}$ by Jia.
- One expects that, for any $h \to \infty$ with $X \to \infty$, the interval $(x, x + h \log x]$ contains primes for almost all $x \in [X, 2X]$.

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- We mostly keep Legos only in one room, so fortunately no analogue with primes in almost all short intervals!!

Primes in almost all short intervals, conditional results

- Conjecturally, for any $h \to \infty$ with $X \to \infty$, the interval $(x, x + h \log x]$ contains primes for almost all $x \in [X, 2X]$.
- Heath-Brown (1982) has established this assuming both RH and the pair correlation conjecture for zeros of $\zeta(s)$.

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- With Jori Merikoski we are working under the unlikely assumption of exceptional characters.

Primes in almost all short intervals, conditional results

Corollary (M.-Merikoski, provisional)

Fix $K > 0$. Assume there is a sequence of moduli $q_j \to \infty$ of primitive quadratic characters χ_j with $q_{j+1} < \exp(\log^K q_j)$ such that for all $j \geq 1$ either

$$L(1, \chi_j) = o\left(\frac{1}{\log q_j}\right)$$

or there exists a real zero β_j of $L(s, \chi_j)$ with

$$1 - \beta_j = o\left(\frac{1}{\log q_j}\right).$$

Let $H/\log X \to \infty$ with $X \to \infty$. Then, for almost all $y \in [X, 2X]$,

$$\sum_{y < p \leq y+H} 1 = (1 + o(1))\frac{H}{\log y}.$$

Almost primes

- One expects that, for any $h \to \infty$ with $X \to \infty$, the interval $(x, x + h \log x]$ contains primes for almost all $x \in [X, 2X]$.
- One can ask similar questions about almost-primes, i.e. P_k numbers that have at most k prime factors or E_k numbers that have exactly k prime factors.

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- Teräväinen has shown that almost intervals of length $(\log X)^{3.51}$ contain an E_2-number. Wu has shown that $(x - x^{101/232}, x]$ contains P_2 numbers for all large x.
- Friedlander and Iwaniec sketched that if $h \to \infty$ with $X \to \infty$, almost all intervals of length $h \log X$ contain P_3-numbers.

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Theorem (M. (202?))

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- There are some almost Legos by other trademarks! I have no experience, so don't know how good approximations they are...

Möbius function

- Let $\mu(n)$ denote the Möbius function

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \cdots p_k \text{ with } p_i \text{ distinct;} \\ 0 & \text{otherwise} \end{cases}$$

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- Now, for $\Re s > 1$,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

so $\mu(n)$ is closely related to $\Lambda(n)$ whose generating Dirichlet series was $-\zeta'/\zeta$.

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so $\mu(n)$ is closely related to $\Lambda(n)$ whose generating Dirichlet series was $-\zeta'/\zeta$.

- In particular

$$PNT \iff \zeta(s) \text{ has no zeros with } \Re s = 1 \iff \sum_{n \leq x} \mu(n) = o(x)$$

$$RH \iff \sum_{n \leq x} \mu(n) = O(x^{1/2+\varepsilon}) \text{ for all } \varepsilon > 0.$$

- Unfortunately, no more Lego analogues!

Möbius in short intervals

- Until 2014 the story for the Möbius function was exactly the same as for $\Lambda(n)$.
- Motohashi and Ramachandra independently adapted Huxley's proof in 1970s to show

$$\sum_{x<n\leq x+H} \mu(n) = o(H), \quad H \geq x^{7/12+\varepsilon}.$$

- Analogously it was known that, for almost all $x \in (X, 2X]$,

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- This almost-all interval result was shown to hold for any $H \to \infty$ with $X \to \infty$ in my work with Radziwiłł (2016). We crucially used that a typical n has prime factors from certain convenient intervals — something that is certainly not true for $n \in \mathbb{P}$.

Möbius in short intervals

A natural question is whether one can do analogous thing for all intervals. Recently Teräväinen and I obtained such a result.

Theorem (M-Teräväinen (202?))

$$\sum_{x < n \leq x+H} \mu(n) = o(H), \quad H \geq x^{0.55+\varepsilon}.$$

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Note that $7/12 = 0.5833\cdots$, and that even under RH one cannot get beyond $1/2$, so we get significantly closer to this natural barrier.

How to prove things related to primes, almost primes or Möbius

- Often there are two steps, a combinatorial and an analytic.
- In combinatorial step combinatorial identity or sieve reduces the problem to so-called type I and type II sums.
- In the analytic step these are estimated.

The combinatorial step

- For instance Vaughan's identity implies that, for any (α_n),

$$\sum_{X<n\leq 2X} \alpha_n \Lambda(n) = \sum_{\substack{X<bc\leq 2X \\ b\leq X^{1/3}}} \alpha_{bc}\mu(b)\log c$$

$$- \sum_{\substack{X<abc\leq 2X \\ b,c\leq X^{1/3}}} \alpha_{abc}\mu(b)\Lambda(c) + \sum_{\substack{X<abc\leq 2X \\ b,c>X^{1/3}}} \alpha_{abc}\mu(b)\Lambda(c).$$

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- Thus $\sum_{X<n\leq 2X} \alpha_n \Lambda(n)$ splits into type I and type II sums

$$\sum_{\substack{X<mn\leq 2X \\ m\leq X^{1/3}}} \alpha_{mn} a_m \quad \text{and} \quad \sum_{\substack{X<mn\leq 2X \\ X^{1/3}\leq m\leq X^{2/3}}} \alpha_{mn} a_m b_n$$

with certain bounded coefficients a_m and b_n.

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with certain bounded coefficients a_m and b_n.

- In type I sums we have a smooth variable n, whereas in type II sums we have genuine bilinear structure.

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- For instance when studying short intervals, one can use Dirichlet polynomials through Perron's formula:

$$\frac{1}{H}\sum_{x<mn\leq x+H} a_m b_n \approx \frac{1}{X}\sum_{X<mn\leq 2X} a_m b_n$$

essentially if

$$\int_{(\log X)^{100}}^{x/H} \left| \sum_{mn\sim X} \frac{a_m b_n}{(mn)^{1+it}} \right| dt = O\left(\frac{x^{1/2}}{(\log x)^{100}}\right).$$

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- Such mean values can be estimated through mean and large value results for Dirichlet polynomials.

Tools in our theorems

- In the problem for almost primes in almost all short intervals, the combinatorial tool is Richert's weighted sieve with β-sieve. This way one reduces to type I sums but can never catch primes.
- In the problem for Möbius in short intervals, in the combinatorial step we use Ramaré's identity and Heath-Brown's identity.
- In the analytic step we use respectively Kloostermania and Dirichlet polynomials.

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Theorem (M. (202?))

Let $h \to \infty$ with $X \to \infty$. Then the interval $(x - h\log X, x]$ contains P_2 numbers for almost all $x \leq X$.

Theorem (M-Teräväinen (202?))

$$\sum_{x < n \leq x+H} \mu(n) = o(H), \quad H \geq x^{0.55+\varepsilon}.$$

Similar method works for other problems. E.g.

Theorem (M-Teräväinen (202?))

$$\sum_{\substack{x < p_1 p_2 \leq x+H \\ p_j \in \mathbb{P}}} 1 = H\frac{\log\log x}{\log x} + O\left(H\frac{\log\log\log x}{\log x}\right), \quad H \geq x^{0.55+\varepsilon}.$$

Thank you!