Embeddings with eulerian faces II: degree conditions

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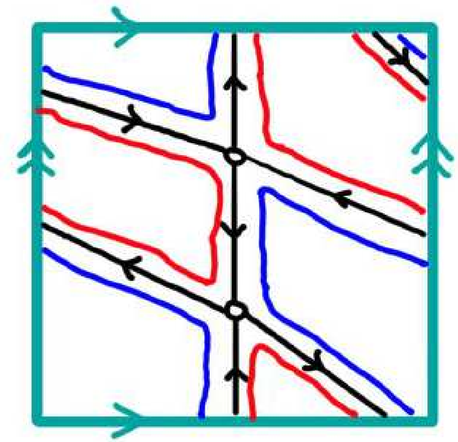
Initial goal: directed OTEF embeddings



Consider directed embeddings of eulerian digraph D: all face boundaries are directed walks.

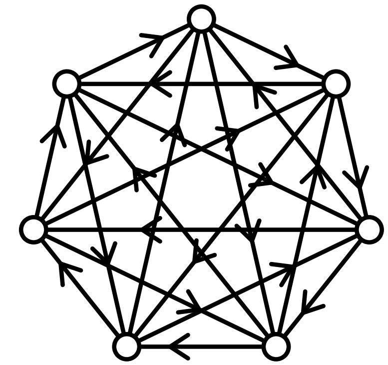
All our embeddings are orientable directed embeddings. Have two face classes: boundary direction clockwise for profaces, anticlockwise for antifaces.

Want maximum genus $\leftrightarrow$ as few faces as possible. Best we could do would be two faces, both bounded by directed euler circuits: Orientable Two Euler Face (OTEF) embedding.

Other motivations: (1) Euler circuit faces give reporter strand walks in DNA model of graph. (2) The euler circuit faces are compatible euler circuits.

Would like to pre-specify euler circuit proface E_0 and find euler circuit antiface. May settle for two antifaces if parity of Euler characteristic requires it.

Previous result: OTEF embeddings for eulerian tournaments



Tournament = orientation of a complete graph; if eulerian
each vertex must have equally many in and out arcs.

Bonnington, Conder, Morton and McKenna (BCMM), 2002: Let D be an eulerian tournament and $n = |V(D)|$ (which must be odd). Let E_0 be a directed euler circuit in D. Then D has an orientable directed embedding with E_0 as the only proface and

(a) one euler circuit antiface, if $n \equiv 3 \pmod 4$ (OTEF embedding), or
(b) two antifaces, if $n \equiv 1 \pmod 4$.

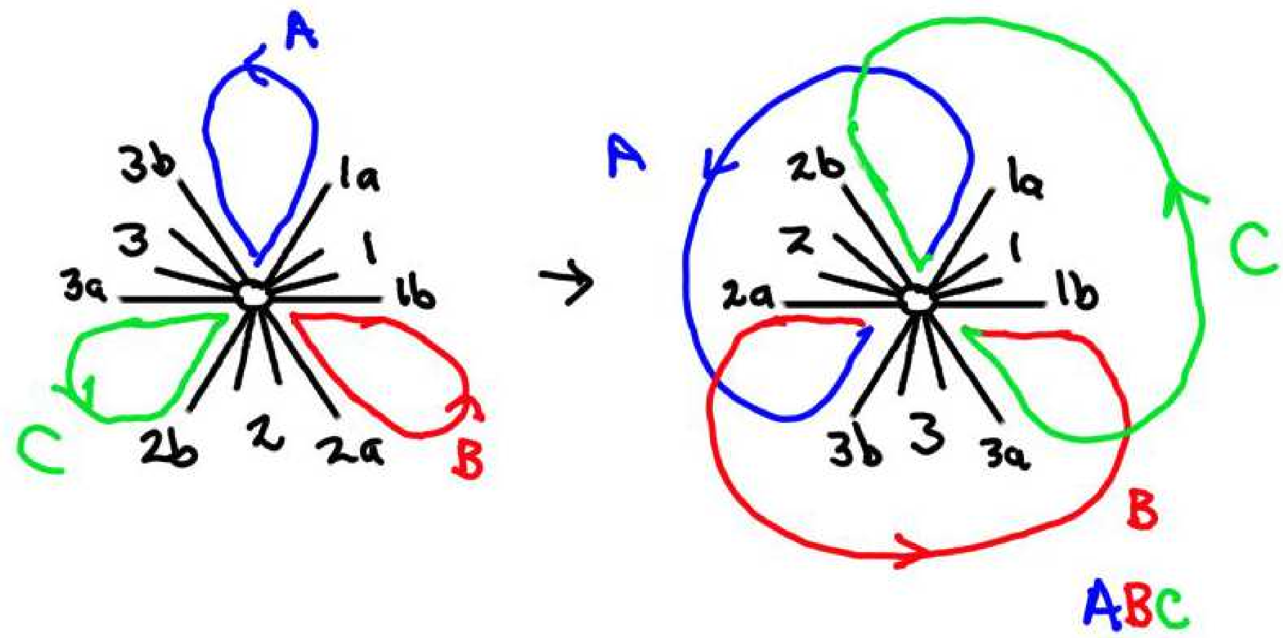
Proof idea: Set up embedding with E_0 as proface and arbitrary antifaces (easy). Then merge antifaces together.

Our approach: Find OTEF embeddings for digraphs with high minimum degree by systematizing and adapting BCMM's proof, and using some new ideas.

Reduction: merging three antifaces

BCMM: If we have three distinct antifaces incident with a vertex v, we can merge them by changing the rotation at v.

Standard operation in undirected case, going back at least to Ringel, 1961.



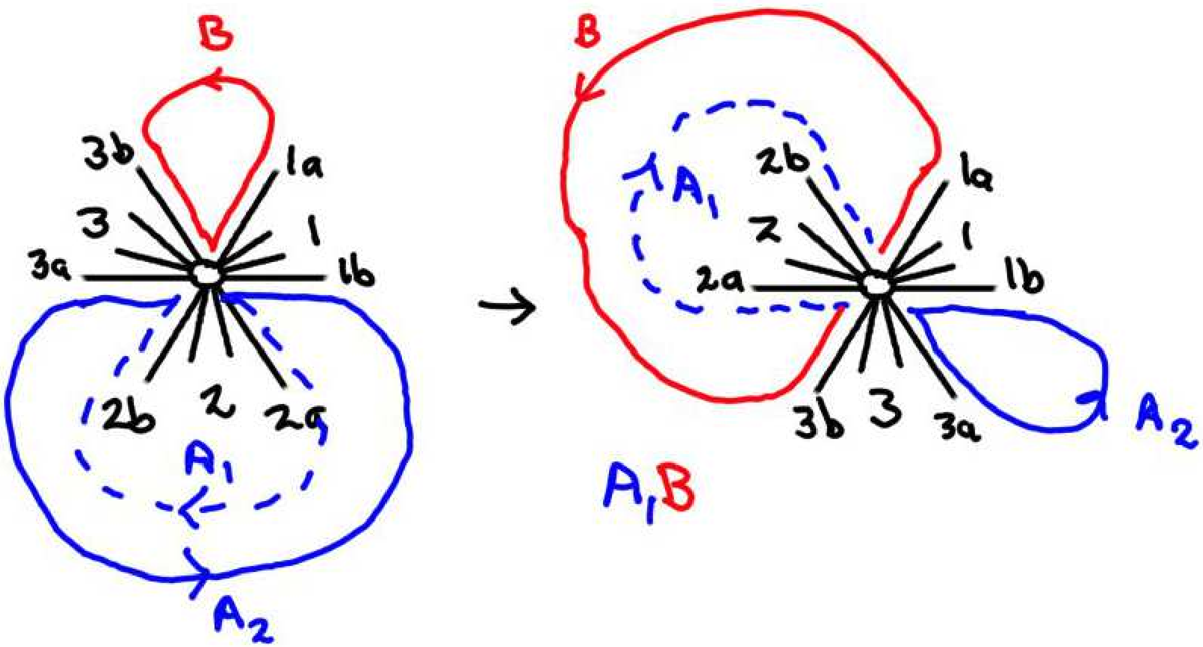
Call vertex reducible if at least three distinct antifaces incident with v.

Embedding is reduced if no reducible vertices: at most two antifaces incident with each vertex. Define vertex to be type P if only incident with face P, and type PQ if incident with faces P and Q.

Recombination: reconfiguring two antifaces

Question: A reduced embedding may still have many antifaces. How do we make progress then? (Cannot combine just two faces: parity issue.)

Lemma: If we have two distinct antifaces A, B incident with a vertex v, where A visits v at least twice, then we can modify the rotation at v to replace A and B by A', B' where $E(B) \subseteq E(B')$, and both $E(A')$ and $E(B')$ intersect $E(A)$.



While recombination does not decrease the number of antifaces, it may create a reducible vertex somewhere.

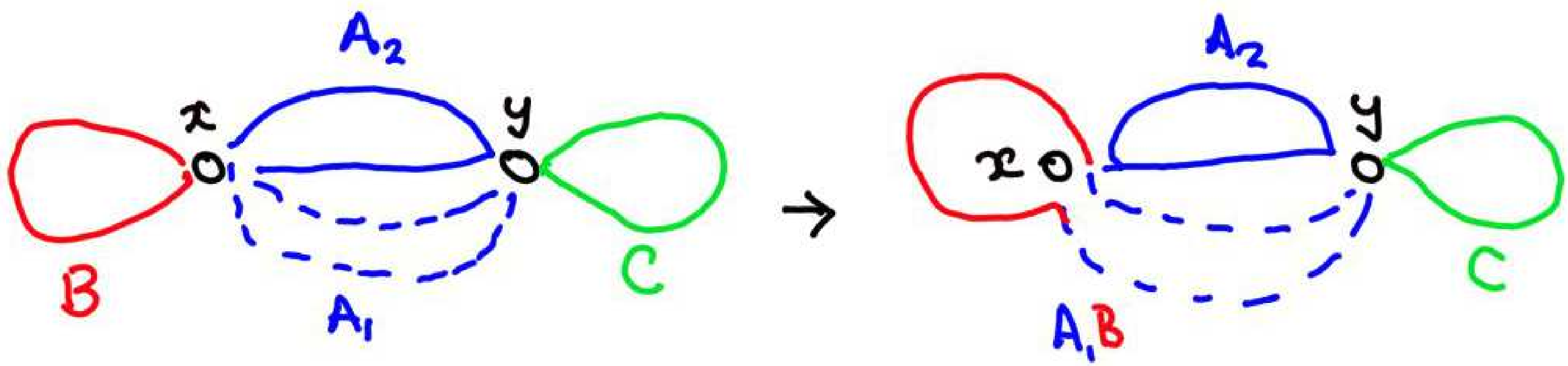
Interlacing $\Rightarrow$ recombine then reduce

Question: How can we ensure that recombination produces a reducible vertex?

Vertices x and y are interlaced on face A if we have $A = (x \ldots y \ldots x \ldots y \ldots)$.

BCMM: If x of type AB and y of type AC are interlaced on A, then we can modify the rotation systems at x and y to merge A, B, and C.

Our proof: Recombining A and B at x makes y a reducible vertex.



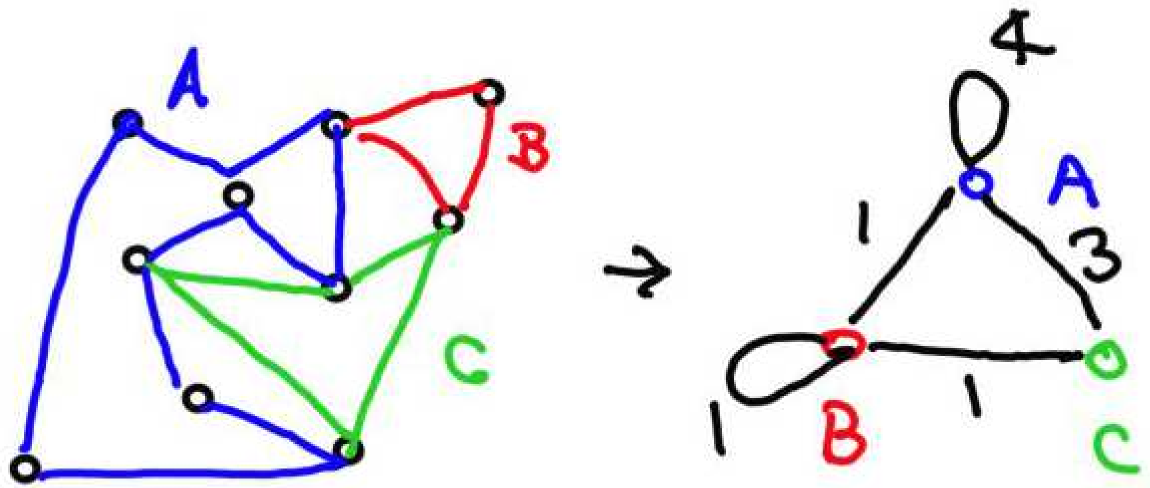
[Picture does not accurately represent rotation systems.]

Graphs of high minimum degree

Definition: We let $k = n - 1 - \delta(G_\mathrm{s})$ where G_s is the underlying undirected simple graph of D. (Every vertex has at most k non-neighbours other than itself.)

To force interlacing. analyze touch graph to restrict face sizes and vertex types.

vertex = face,
edge = vertex type,
edge weight = number of vertices of that type.



Face Size Lemma: If $n \geq 6k + 1$ and we have a reduced embedding, then

(a) there is a face A with $|V(A)| \geq n - k$, or

(b) all but one or two faces P have $|V(P)| \leq k$, or

(c) there are exactly three vertex types AB, AC and BC.

Blow Up Lemma

Face Size Lemma: If $n \geq 6k+1$ and we have a reduced embedding, then

(a) there is a face A with $|V(A)| \geq n-k$, or

(b) all but one or two faces P have $|V(P)| \leq k$, or

(c) there are exactly three vertex types AB, AC and BC.

For BCMM with tournaments, have $k = 0$, just get cases (a) and (c). That, plus fact that underlying graph is complete, is enough to force interlacing.

However, with noncomplete graphs we need faces of some minimum size to force interlacing.

Blow Up Lemma: Suppose v is a vertex of type PQ and v is joined to at least $d \geq 3$ distinct vertices (other than itself) by edges of P. Then we can recombine P, Q to produce P', Q' with $|V(P')|, |V(Q')| \geq \lceil d/2 \rceil$. Choose segments of P for recombination with about $d/2$ neighbours each.

Blow Up Corollary: Suppose that $n \geq 9k+10$ and v is of type PQ. Then we can replace P, Q by P', Q' with $|V(P')|, |V(Q')| \geq 2k+3$.

Existence of OTEFs

E & EM, 2021+: Suppose D is an eulerian digraph with euler circuit E_0 and $\delta(G_{\rm s}) \geq (8n+1)/9$ ($n \geq 9k+10$) where $G_{\rm s}$ is the underlying undirected simple graph. Then D has an orientable directed embedding with E_0 as the only proface and one euler circuit antiface (an OTEF embedding) or two antifaces.

Proof outline: Start with arbitrary embedding with E_0 as only proface.

while more than 2 antifaces $\{$

if reduced, apply Blow Up Corollary: two faces with $\geq 2k+3$ vertices;

if reduced, apply Blow Up Corollary again: three faces with $\geq 2k+3$ vertices;

if reduced

if (a) A with $|V(A)| \geq n-k$

can find interlacing and create reducible vertex;

else (c) there must be exactly three vertex types AB, AC, BC

can find interlacing and create reducible vertex;

reduce until no reducible vertices;

$\}$

Situation (b) from Face Size Lemma does not occur, due to blow ups.

Finding interlacing

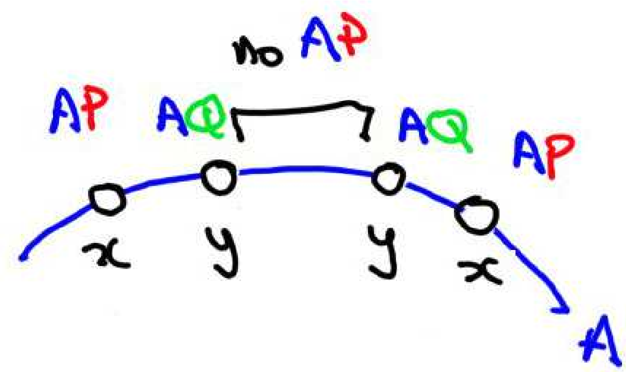
Will outline argument for interlacing for Face Size Lemma (a); (c) is similar.

Using Face Size Lemma (a) and blow ups:

- By (a) have $|V(A)| \geq n-k$, and by blowing up $|V(B)|, |V(C)| \geq 2k+3$.
- Thus, at least $k+3$ vertices of type AB and of type AC.
- Thus, each AB vertex adjacent (via edge of A) to at least 3 AC vertices, and vice versa. Hence each AB or AC vertex appears at least twice on A.

Now argument from BCMM:

- Choose interval $a_0 a_1 \ldots a_\ell$ along A so that
  (1) $a_0 = a_\ell = x$ is of type AP, $P \in \{B, C\}$;
  (2) some $a_i = y$, $1 \leq i \leq \ell - 1$ is of type AQ, $\{P, Q\} = \{B, C\}$;
  (3) subject to (1) and (2), ℓ is minimum.



- Then x and y are interlaced on A: No AP neigbours between first and last y's in $a_0 \ldots a_\ell$ by (3), and then only two AP neighbours for occurrences of y in $a_0 \ldots a_\ell$, so some occurrence of y outside of $a_0 \ldots a_\ell$.

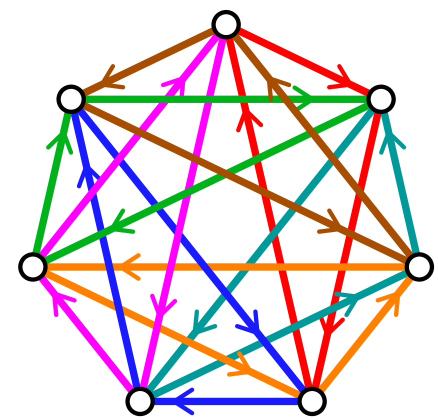
Directed embeddings with specified profaces

Observation: Our argument (or the BCMM argument) does not use the proface E_0 anywhere. So really we can specify any profaces we like.

E & EM, 2021+: Suppose D is an eulerian digraph with $\delta(G_s) \geq (8n+1)/9$ where G_s is the underlying undirected simple graph. Let $\mathcal{W}$ be an arbitrary set of edge-disjoint closed directed walks in D. Then D has an orientable directed embedding with $\mathcal{W}$ as a subset of the profaces and either one euler circuit antiface or two antifaces.

This gives maximum genus directed embeddings subject to specified profaces. For example, the following is a corollary of our result.

Griggs, McCourt & Širáň, 2020: Decompose the edges of K_n into triangles (a Steiner triple system), and orient each triangle, giving digraph D. Then there is an orientable directed embedding of D whose faces are the oriented triangles, plus one euler circuit face.



Future work

Main goal: Extend conditions under which we can find OTEF directed embeddings, or even directed embeddings with specified profaces and one euler circuit antiface.

- It would be nice to obtain a general result which covers the following:

Griggs, McCourt & Širáň, 2020: Decompose the edges of $K_{n,n,n}$ (n odd) into triangles (a Latin square of odd order), and orient each triangle, giving digraph D. Then there is an orientable directed embedding of D whose faces are the oriented triangles, plus one euler circuit face.

A minimum degree result that covers this would need to work for $\delta(G_{\mathrm{s}}) \geq 2n/3$.

- Perhaps it suffices to have a 4-edge-connected eulerian digraph with minimum degree 6? We know that minimum degree 4 is not enough.

Thank you!