QUANTUM GROUPS IN THE HEAT

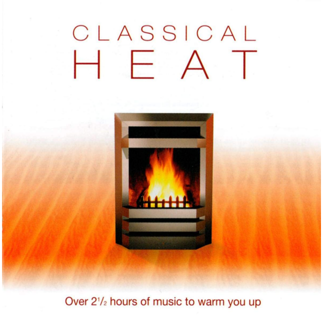
Amaury FRESLON, Lucas TEYSSIER & Simeng WANG

université PARIS-SACLAY

Département de Mathématiques d'Orsay

Classical heat

What happens if we heat $SO(N)$?



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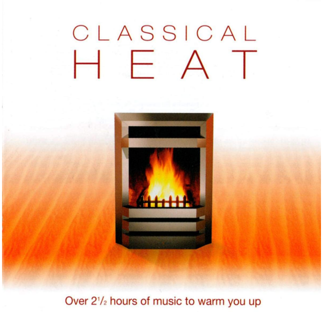
What happens if we heat $SO(N)$?

- Simple connected compact Lie group ;
- Canonical Riemanian structure ;
- Laplace-Beltrami operator Δ ;
- Heat semi-group $\left(e^{-\Delta t}\right)_{t\in\mathbb{R}_+}$.



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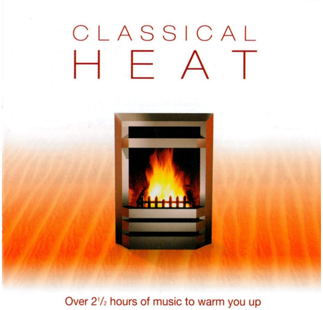
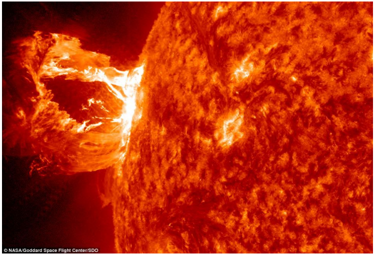


Theorem (Méliot)

The heat semi-group spreads uniformly on $SO(N)$ in time $\ln(N)$.

Quantum heat

What happens if we heat the *quantum orthogonal group* O_N^+ ?



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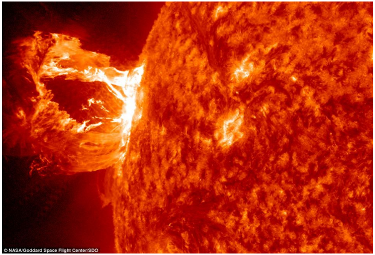

- $\mathcal{O}(O_N^+) = \left\langle u_{ij} = u_{ij}^*, 1 \leqslant i,j \leqslant N \quad \middle| \quad \sum_{k=1}^{N} u_{ik}u_{jk} = \delta_{ij} = \sum_{k=1}^{N} u_{ki}u_{kj} \right\rangle$;
- $\delta : u_{ij} \to \sum_{k=1}^{N} u_{ik} \otimes u_{kj}$.

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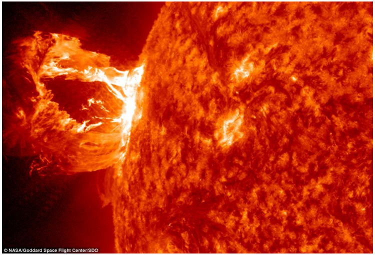

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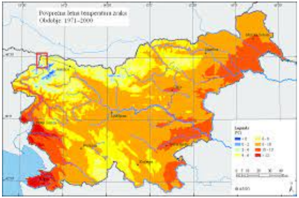
Problem : No "canonical" Riemanian structure ...

Probabilistic heat

Let $(X_t)_{t\in\mathbb{R}_+}$ be a *Lévy process* on $SO(N)$, i.e.

- $X_{t_2}X_{t_1}^{-1}, \cdots, X_{t_n}X_{t_{n-1}}^{-1}$ are independent for $t_1 < \cdots < t_n$;
- $\mathrm{Law}\left(X_t X_s^{-1}\right) = \mathrm{Law}(X_{t-s})$ for $t \geqslant s$;
- $X_t \to X_0$ when $t \to 0$.



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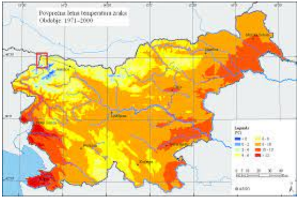

Theorem (Liao)

If $(X_t)_{t\in\mathbb{R}_+}$ is conjugation-invariant, then

$$L = b.\Delta + \textit{Lévy},$$

where L is the infinitesimal generator and $b \in \mathbb{R}_+$.

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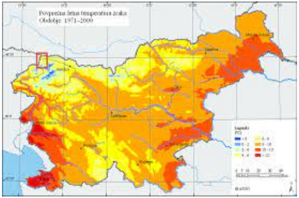

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$$L(f) = \lim_{t\to 0} \frac{\mu_t * f - f}{t}$$

where $\mu_t = \mathrm{Law}(X_t)$.

Quantum probabilistic heat

Ad-invariant Lévy process on O_N^+ :

- Convolution semi-group of states φ_t : $\mathcal{O}(O_N^+) \to \mathbb{C}$;
- Infinitesimal generator $L = \mathscr{L} \circ \mathbb{E}_{\text{central}}$;
- $\mathscr{L} : \mathbb{C}[-N, N] \to \mathbb{C}[-N, N]$.

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Theorem (Cipriani–Franz–Kula)

We have

$$\mathscr{L}(P_n) = b\frac{P_n'(N)}{P_n(N)} + \text{Lévy}.$$

where $b \in \mathbb{R}_+$.

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Dilated Чебышёв polynomials of the second kind : $P_0(X) = 1$, $P_1(X) = X$,

$$XP_n(X) = P_{n+1}(X) + P_{n-1}(X).$$

Mixing time

Theorem (Freslon-Teyssier-Wang)

Let $t_N = N\ln(N)$*, then for all* $\epsilon > 0$*,*

$$\lim_{N\to+\infty} \|\varphi_{(1-\epsilon)t_N} - \mathrm{Haar}\|_{\mathscr{O}\left(O_N^+\right)^*} = 1$$

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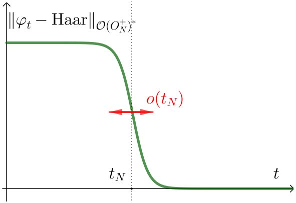
This is the *cutoff phenomenon* :



Limit profile

We can zoom in on the threshold :

Theorem (F.-Teyssier-Wang)

Set $t_N = N\ln(N)/2$. *Then, for any* $c > 0$

$$\frac{1}{2}\lim_{N\to+\infty}\left\|\varphi_{t_N+cN} - \mathrm{Haar}\right\|_{\mathscr{O}\left(O_N^+\right)^*} = \left\|\mathrm{Poiss}^+(e^{2c}, -e^{-c}) \boxplus \delta_{e^c+e^{-c}} - \mathrm{SC}\right\|_{TV}$$
$$= \left\|\mathrm{Meix}^+(-e^{-c}, 0) \boxplus \delta_{e^{-c}} - \mathrm{Meix}^+(0,0)\right\|_{TV}$$

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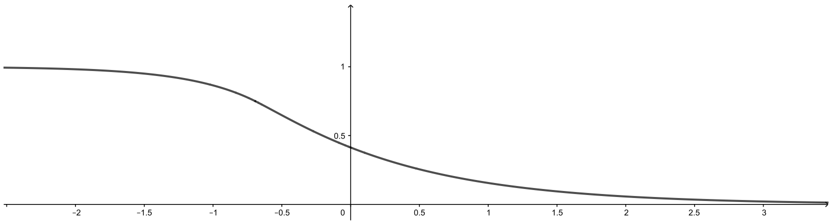
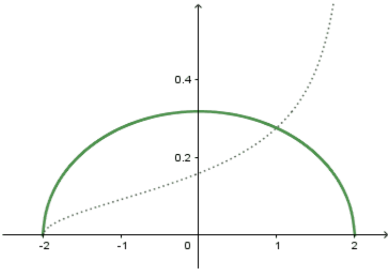


A look at the proof

- For $c > 0$, $\varphi_{N\ln(N)+cN}$ is absolutely continuous with respect to Haar,



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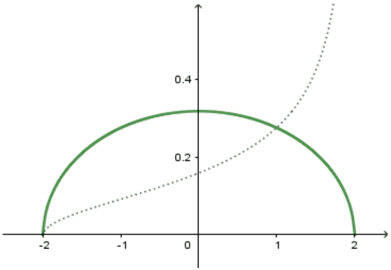
$$\left\|\varphi_{t_N+cN} - \mathrm{Haar}\right\|_{\mathcal{O}(O_N^+)^*} = \frac{1}{2}\left\|\sum_{n=1}^{+\infty} P_n(N) e^{-(t_N+cN)\frac{P_n'(N)}{P_n(N)}} P_n(\cdot)\right\|_1$$

$$\leqslant \frac{1}{2}\sqrt{\sum_{n=1}^{+\infty} e^{-2(t_N+cN)\frac{P_n'(N)}{P_n(N)}} P_n(N)^2}$$



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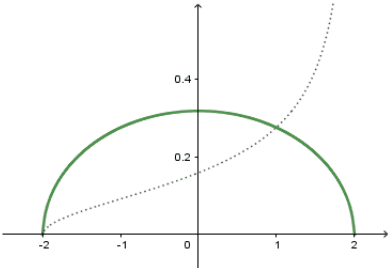
- **Interpretation :** $\mathrm{SC} = \mathrm{Law}_h(\chi_1)$ and for O_N, $\mathrm{Law}_h(\chi_1) = \mathrm{Law}_h(\mathrm{Tr})$.



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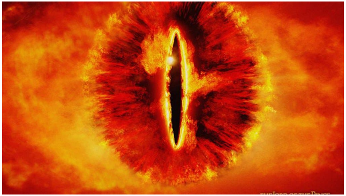
- For $c < 0$, things get … wild



$$m_t^{(N)} = \alpha(t)\delta_{\widetilde{N}(t)} + \sum_{n=0}^{+\infty}\left[e^{-t\frac{P_n'(N)}{P_n(N)}}P_n(N) - \alpha(t)P_n\left(\widetilde{N}(t)\right)\right]P_n \mathrm{dSC}$$

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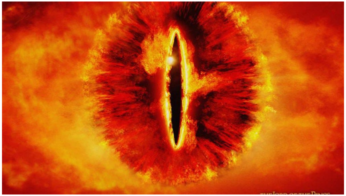
- $\int_{-N}^{N} f(x)\mathrm{d}m_t^{(N)}(x) = \varphi_{t|\mathcal{O}\left(O_N^+\right)_{\text{central}}}(f)$



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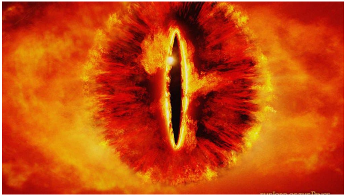
- For $c < 0$, things get ... wild
- $\int_{-N}^{N} f(x)\mathrm{d}m_t^{(N)}(x) = \varphi_{t|\mathscr{O}\left(O_N^+\right)_{\text{central}}}(f)$
- We can find $\widetilde{N}(t) \in [-N, N]$ such that



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Thanks for your attention 😎