Hirzebruch-type inequalities and extreme point-line configurations

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Our agenda for today:

1. Hirzebruch-type inequalities – what does it really mean?
2. Examples and techniques
3. Interesting applications

The Sylvester Problem has been posed by James Joseph Sylvester in 1893 in Educational Times.

Let n given points have the property that the line joining any two of them passes through a third point of the set. Must the n points all lie on one line?

The story is quite complicated, as we know the answer to the above problem was given by Tibor Gallai (aka Tibor Grünwald) around 1940, announced by P. Erdős in 1943, and finally in print around 1944.

Theorem (Gallai)

In any finite configuration of n points in the plane, not all one a line, there is a line which contains exactly two of the points.

Definition

Let $\mathcal{P}$ be a finite set of mutually distinct points in the real plane. A line ℓ passing through exactly points from the set $\mathcal{P}$ is called an ordinary line. If there exists a line ℓ' passing through exactly r points from $\mathcal{P}$, then we call such a line as r-rich.

It is clear that Sylvester's problem has a positive solution if we replace the real plane by the real projective plane.
Using the duality we can rephrase Sylvester's problem on points and ordinary lines to arrangements of lines and intersection points. Let $\mathcal{L} = \{H_1, ...H_d\} \subset \mathbb{P}^2_{\mathbb{R}}$ be an arrangement of d lines, and we will denote by $t_r(\mathcal{L}) = t_r$ the number of r-fold points, i.e., points in the plane where exactly r lines from $\mathcal{L}$ meet.

Theorem (Melchior, 1944)

Let $\mathcal{L} = \{H_1, ...H_d\} \subset \mathbb{P}^2_{\mathbb{R}}$ be an arrangement of $d \geqslant 3$ lines which is not a pencil. Then

$$t_2 \geqslant 3 + \sum_{r \geqslant 4} (r-3) t_r.$$

The equality holds iff $\mathcal{L}$ is a simplicial line arrangement.

Comments

- Melchior's proof is based on the classical Euler formula. Every line arrangement of lines $\mathcal{L}$ determines a cellular decomposition of $\mathbb{P}^2_{\mathbb{R}}$ giving v vertices, e edges, and f faces, so we have

$$v - e + f = \chi(\mathbb{P}^2_{\mathbb{R}}).$$

- The classification of extreme cases for Melchior's inequality is not completed, there is no complete classification of simplicial line arrangements, somehow surprisingly.

It was very natural to ask whether Melchior's inequality holds over different fields of definition. In positive characteristic, it is enough to consider Fano's plane, over the complex numbers we can take the dual Hesse arrangement of 9 lines and 12 triple intersection points. However, Iitaka in 1979 claimed that Melchior's inequality holds over the reals. It is worth pointing out that Iitaka's paper has nothing to do with the combinatorics of line arrangements in the plane, he wanted to understand the so-called geography problem for open surfaces constructed with use line arrangements.

Theorem (Hirzebruch, 1983)

Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $d \geqslant 6$ lines such that $t_d = t_{d-1} = t_{d-2} = 0$, and let $n \geqslant 2$. Then

$$P_{\mathcal{L}}(n) = n^2(f_0 - d) + 2n(d - f_1 + f_0) + 2f_1 + f_0 - d - 4t_2 \geqslant 0,$$

where $f_0 = \sum_{r \geqslant 2} t_r$ and $f_1 = \sum_{r \geqslant 2} rt_r$.
In particular, for $n = 3$ one has the following inequality

$$t_2 + t_3 \geqslant d + \sum_{r \geqslant 5} (r-4)t_r.$$

1. Hirzebruch's construction is quite technical, but the main backbone is, let's say, meta-the-same as in Melchior's approach. In order to get his inequality, Hirzebruch constructed an abelian cover branched along a given line arrangement. Then he computes the Chern numbers (viewed as a natural generalization of the Euler characteristic), and the last step is to use the Bogomolov-Miyaoka-Yau inequality. A mild-explanation is presented in my recent survey.
2. Let us consider $\mathcal{A}_1(6)$ arrangement of $d = 6$ lines, and $t_2 = 3, t_3 = 4$. Then we can compute the Hirzebruch polynomial

$$P_{\mathcal{A}_1(6)}(n) = n^2 - 10n + 25.$$

It means that for $n = 5$ we get $P_{\mathcal{A}_1(6)}(5) = 0$. This has an extremely important geometric meaning.

Now we pass to the natural problem.

Shall we classify all line arrangements $\mathcal{A}$ in the complex projective plane such that there is $m \in \mathbb{Z}_{\geqslant 2}$ with $P_{\mathcal{A}}(m) = 0$?

This is exactly the same question as providing a complete characterization of line arrangements in $\mathbb{P}^2_{\mathbb{R}}$ for which the condition $t_2 = 3 + \sum_{r \geqslant 4}(r-3)t_r$ holds.

For $n = 5$, one of arrangements $\mathcal{A}$ with $P_{\mathcal{A}}(5) = 0$ is exactly $\mathcal{A} = \mathcal{A}_1(6)$. For $n = 3$ the only one arrangement $\mathcal{A}$ with $P_{\mathcal{A}}(3) = 0$ is the Hesse arrangement of 12 lines with $t_4 = 9$ and $t_2 = 12$. This is a very remarkable result!

Open Problem

For $n = 2$, Höfer in his PhD thesis was able to find a combinatorial type of line arrangements for which we can potentially get $P_{\mathcal{A}}(2) = 0$.

Problem

Is it possible to construct arrangements of $d = 12m + 3$ lines in the complex projective plane such that $t_2 = 12m^2 + 15m + 3$, $t_6 = 4m^2 + m$ with $m \in \mathbb{Z}_{\geqslant 3}$?

It is clear that we cannot construct such arrangements over the real numbers, and also as pseudolines.

Theorem (Bojanowski & Langer, 2002)

Let $\mathcal{L} = \{\ell_1, ..., \ell_d\} \subset \mathbb{P}^2_{\mathbb{C}}$ *be a line arrangement with* $d \geqslant 6$ *such that* $t_r = 0$ *for* $r > \frac{2d}{3}$. *Then*

$$t_2 + \frac{3}{4}t_3 \geqslant d + \sum_{r \geqslant 5} \left(\frac{r^2}{4} - r \right) t_r. \tag{1}$$

Remarks

Let us now list examples of line arrangements for which we obtain equality in (1) – our list is probably far from complete.

1. *Icosahedron arrangement* consisting of 15 lines and $t_2 = 15, t_3 = 10, t_5 = 6$.
2. *Ceva's arrangements* consisting of $3n$ lines $(n \geqslant 4)$, and $t_3 = n^2, t_n = 3$.
3. *The extended Ceva's arrangements* consisting of $3n + 3$ lines with $n \geqslant 3$, and $t_2 = 3n$, $t_3 = n^2$, $t_{n+2} = 3$.
4. *The Hesse arrangement* consisting of 12 lines and $t_4 = 9, t_2 = 12$.
5. The union of Ceva's arrangement of 9 lines and the Hesse arrangement consisting of $d = 12 + 9$ lines with $t_2 = 36$, $t_4 = 9$, $t_5 = 12$.
6. *Klein's arrangement* consisting of 21 lines and $t_3 = 28, t_4 = 21$.
7. *Wiman's arrangement* consisting of 45 lines and $t_3 = 120, t_4 = 45, t_5 = 36$.

We start with the Weak Dirac Conjecture.

Conjecture (Dirac)

Every set $\mathcal{P}$ of n non-collinear points contains a point in at least $\frac{n}{2}$ lines determined by $\mathcal{P}$.

It turned out that the Dirac conjecture is false – the smallest counterexample has $n = 7$ points, namely the vertices of a triangle together with the midpoints of its sides and its centroid. However, the conjecture was resolved positively by Green and Tao for very large n. In this view, we can formulate the actual Dirac conjecture which is, according to our best knowledge, open.

Conjecture

There is a constant c such that every set $\mathcal{P}$ of n non-collinear points contains a point in at least $\frac{n}{2} - c$ lines determined by $\mathcal{P}$.

In 1961, P. Erdős proposed the following Weak Dirac Conjecture.

Conjecture (WDC)

Every set $\mathcal{P}$ of n non-collinear points in the plane (presumably over the real numbers) contains a point which is incident to at least $\lceil \frac{n}{c} \rceil$ lines determined by $\mathcal{P}$, for some constant $c > 0$.

The Weak Dirac Conjecture was proved independently by Beck and Szemerédi-Trotter, but they did not specify the actual value of c. In 2012, Payne and Wood showed the WDC with $c = 37$, and one of the main ingredients of their proof is Hirzebruch's inequality.

Theorem (Han)

The Weak Dirac Conjecture holds with $c = 3$.

Theorem (Beck)

For a finite set $\mathcal{P}$ of n points in $\mathbb{R}^2$ one of the following is true:

- *there exists a line that contains $c_1 n$ points from $\mathcal{P}$ for some positive c_1;*
- *there are at least $c_2 n^2$ lines determined by $\mathcal{P}$.*

Beck in his paper gave $c_1 = \frac{1}{100}$ and c_2 was unspecified, Payne and Wood in provided $c_1 = c_2 = \frac{1}{100}$. Using Langer's inequality, Frank de Zeeuw observed that one can significantly improve estimations on c_1 and c_2.

Theorem (de Zeeuw)

Let $\mathcal{P}$ be a finite set of n points in $\mathbb{R}^2$, then one of the following is true:

- *there is a line that contains more than $\frac{6+\sqrt{3}}{9}n$ points of $\mathcal{P}$;*
- *there are at least $\frac{n^2}{9}$ lines determined by $\mathcal{P}$.*

The final word:

"One should never try to prove anything that is not almost obvious." — Alexander Grothendieck

References

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