Factorizations of infinite graphs

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Joint work with Tommaso Traetta

The Factorization Problem

Let Λ be a graph of order $\aleph$ and let $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$ be a family of (non-empty) infinite graphs, each of which has order $\aleph$.

Problem

The Factorization Problem $FP(\mathcal{F}, \Lambda)$ *asks for a factorization, that is a decomposition into spanning subgraphs,* $\mathcal{G} = \{\Gamma_\alpha : \alpha \in \mathcal{A}\}$ *of* Λ *such that* Γ_α *is isomorphic to* F_α*, for every* $\alpha \in \mathcal{A}$.

- If Λ is the complete graph of order $\aleph$, we simply write $FP(\mathcal{F})$.
- If in addition each F_α is isomorphic to a given graph F and $|\mathcal{A}| = \aleph$, we write $FP(F)$.

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Theorem (Kőhler 1977)

There exists a (cyclic) resolvable 2-$(v, k, 1)$ *design whenever* $v = \aleph_0$ *and* k *is finite.*

Theorem (Danziger, Horsley, Webb 2014)

Every infinite 2-$(v, k, 1)$ *design with* $k < v$ *is necessarily resolvable.*

Theorem (Bonvicini, Mazzuoccolo 2010; SC, Traetta 2021^{+})

There exists a G*-regular 1-factorization of* $K_{\aleph}$ *whenever* $|G| = \aleph$*.*

Theorem (SC 2020)

Let F *be a graph whose order is the cardinal number* $\aleph$*.* $FP(F)$ *has a* G*-regular solution whenever the following two conditions hold:*

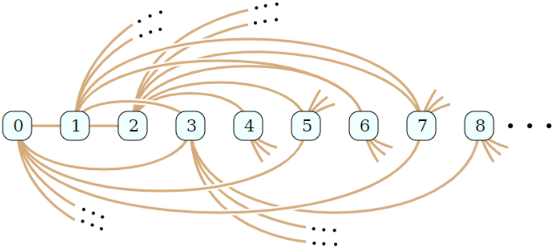
- F *is locally finite;*
- G *is an involution free group of order* $\aleph$*.*

The Rado graph

Definition

The Rado graph R, is defined as follows:

- $V(R) = \mathbb{N}$;
- *a pair* $\{i, j\}$ *with* $i < j$ *is an edge of R if and only if the i-th bit of the binary representation of j is one.*



Known results for the Rado graph

Theorem (Cameron 1997)

Let $\mathcal{F}$ be a countable family of locally finite countable graphs. Then $FP(\mathcal{F}, R)$ has a solution.

Lemma

Let F be a locally finite countable graph. Then, for any factor Γ of R that is isomorphic to F, $R \setminus \Gamma$ is isomorphic to R.

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Factorizations of the Rado graph

The domination number of a graph is defined as the minimum cardinality of a dominating set.

Theorem (SC, Traetta 2021$^+$)

Let $\mathcal{F}$ be a countable family of countable graphs. Then FP($\mathcal{F}$, R) has a solution if and only if the domination number of each graph of $\mathcal{F}$ is infinite.

Lemma

Let F be a countable graph whose domination number is infinite. Then there exists a factor Γ of R such that:

- *Γ is isomorpic to F;*
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From R to $K_{\mathbb{N}}$

Corollary

Let $\mathcal{F}$ be a countable family of countable graphs. $FP(\mathcal{F})$ has a solution whenever the domination number of each graph in $\mathcal{F}$ is infinite.

Proof.

- R is self-complementary and thus there exist R_1 and R_2 that are copies of R which factorize $K_{\mathbb{N}}$.
- We partition $\mathcal{F}$ into two countable families $\mathcal{F}_1$ and $\mathcal{F}_2$.
- Because of the previous theorem there is a solution $\mathcal{G}_i$ to $FP(\mathcal{F}_i, R_i)$, for $i = 1, 2$.
- $\mathcal{G}_1 \cup \mathcal{G}_2$ provides a solution to $FP(\mathcal{F})$.

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Generalized Rado graphs

For $\aleph = |\mathbb{N}|$, the following property characterizes R.

- $\star_\aleph$ For every disjoint sets of vertices U and W whose cardinality is smaller than $\aleph$, there exists a vertex z adjacent to all the vertices of U and non-adjacent to all the vertices of V.

Proposition (Cameron 1997)

Any two graphs of order $\aleph$ that satisfy property $\star_\aleph$ are pairwise isomorphic.

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Does a graph $R_\aleph$ that satisfies $\star_\aleph$ exist?

It exists for any infinite cardinal $\aleph$ if and only if we assume the generalized continuum hypothesis!

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Existence results

Theorem (SC, Traetta 2021$^+$)

Let $\mathcal{F}$ be a family of graphs, each of which has order $\aleph$. FP($\mathcal{F}$) has a solution whenever the following two conditions hold:

1) $|\mathcal{F}| = \aleph$;
2) *the domination number of each graph in $\mathcal{F}$ is $\aleph$.*

The proof is based on the following methods:

- Transfinite induction (it requires the axiom of choice).
- A variation of the Cantor back and forth method.

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From $K_\aleph$ to $R_\aleph$

Let $\aleph$ be such that $R_\aleph$ exists.

Theorem (SC, Traetta 2021$^+$)

Let $\mathcal{F}$ be a family of graphs of order $\aleph$ and $|\mathcal{F}| = \aleph$. FP$(\mathcal{F}, R_\aleph)$ has a solution if and only if the domination number of each graph in $\mathcal{F}$ is $\aleph$.

Proof.

- Since the domination number of the $\aleph$-Rado graph is $\aleph$, the same must hold for each graph of $\mathcal{F}$.

Now we prove that this condition is also sufficient.

- $\mathcal{F}' := \mathcal{F} \cup \{R_\aleph\}$ satisfies the hypothesis of previous theorem.
- Therefore $FP(\mathcal{F}')$ admits a solution and $\mathcal{F}$ factorizes $K_\aleph \setminus R_\aleph$.
- Since $R_\aleph$ is self-complementary, $\mathcal{F}$ factorizes the $\aleph$-Rado graph.

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Necessity of condition 2?

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Is it necessary to require that the domination number of each graph in $\mathcal{F}$ is $\aleph$ to have a solution of FP$(\mathcal{F})$?

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The k-star problem

Definition

Let us consider the k-star S_k, defined as follows.

- *the star S_1 is the graph with vertex-set $\mathbb{N}$ and whose edges are of the form $\{0, i\}$ for every $i \in \mathbb{N} \setminus \{0\}$;*
- *the k-star S_k is the vertex-disjoint union of k stars.*

Note that S_k has a dominating set of size k.

Theorem (SC, Traetta 2021$^+$)

$FP(S_k)$ has no solution for $k \in \{1, 2\}$ but it has whenever $k \geq 4$.

The case $k = 3$ is left open.

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References

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