A reduction of the spectrum problem for sun systems

Anita Pasotti
anita.pasotti@unibs.it

Università degli Studi di Brescia, Italy

Joint work with Marco Buratti and Tommaso Traetta

- M. Buratti, A. Pasotti, T. Traetta, *A reduction of the spectrum problem for odd sun systems and the prime case*, J. Combin. Des. **29** (2021), 5–37.

- A. Pasotti, T. Traetta, *Even sun systems of the complete graph*, in preparation.

Unicyclic graphs

Definition

$(t_1, t_2, \ldots, t_k)$-unicycle $=$ k-cycle $(v_1, v_2, \ldots, v_k)$ with t_i pendant edges attached to v_i, for $1 \leq i \leq k$.

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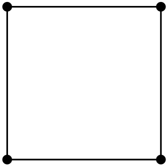

Figure: (0,0,0,0)-unicycle

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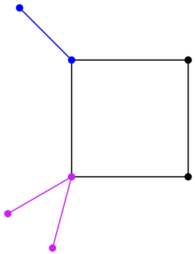

Figure: (1,0,0,2)-unicycle

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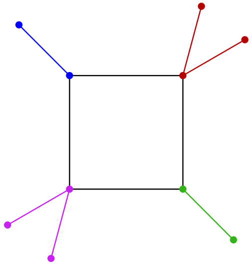

Figure: (1,2,1,2)-unicycle

General problem

Problem [M. Buratti (2018)]

Given a sequence $(t_1, t_2, \dots, t_k)$ of nonnegative integers, find the set of all values v for which there exists a decomposition of K_v into copies of the $(t_1, t_2, \dots, t_k)$-unicycle.

Cycle systems

$\underbrace{(0,0,\ldots,0)}_{k}$-unicycle $= C_k =$ cycle of length k

Trivial necessary conditions

If there exists a k-cycle system of K_v then

- $v \geq k$;
- v odd;
- $v(v-1) \equiv 0 \pmod{2k}$.

$k \leq v < 3k$

Theorem

There exists a k-cycle system of K_v for every admissible v if and only if there exists a k-cycle system of K_v for every admissible v with $k \leq v < 3k$.

- k even, J.C. Bermond, C. Huang, D. Sotteau (1978);
- k odd, D.G. Hoffman, C.C. Lindner, C.A. Rodger (1989).

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Theorem

Let $k \geq 3$. There exists a k-cycle system of K_v for every admissible v.

- k even, M. Sajna (2002);
- k odd, B. Alspach, H. Gavlas (2001);
  M. Buratti (2003).

Sun systems

$\underbrace{(1,1,\ldots,1)}_{k}$-unicycle $= S_k =$ sun of length k

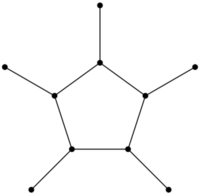

Figure: $(1,1,1,1,1)$-unicycle $= S_5$

Sun systems

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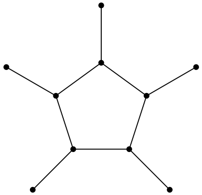

Figure: $(1,1,1,1,1)$-unicycle $= S_5$

Trivial necessary conditions

If there exists a k-sun system of K_v then

- $v \geq 2k$;
- $v(v-1) \equiv 0 \pmod{4k}$.

Known results and conjecture

There exists a k-sun system of K_v, for every admissible v, when

- $k = 3$, C.M. Fu, N.H. Jhuang, Y.L. Lin, H.M. Sung (2012);
- $k = 5$, C.M. Fu, M.H. Huang, Y.L. Lin (2013);
- $k = 4, 6, 8$, Z. Liang, J. Guo (2010);
- $k = 10, 14, 2^t$, C.M. Fu, N.H. Jhuang, Y.L. Lin, H.M. Sung (2012).

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Conjecture 1 [M. Buratti, A.P., T. Traetta (2021)]

Let $k \geq 3$. There exists a k-sun system of K_v if and only if the trivial necessary conditions hold.

Main results

Theorem

Let $k \geq 3$. Conjecture 1 is true if and only if there exists a k-sun system of K_v for every admissible v with $2k < v < 6k$.

- k odd, M. Buratti, A.P., T. Traetta (2021);
- k even, A.P., T. Traetta (202?).

Main results

Theorem [M. Buratti, A.P., T. Traetta (2021)]

For every odd prime k, there exists a k-sun system of K_v for every admissible v.

Theorem [A.P., T. Traetta (202?)]

For every prime k, there exists a $2k$-sun system of K_v for every admissible v.

It is sufficient to construct...

Let $k \geq 3$ be a prime.

It is sufficient to construct a k-sun system of K_v for

- $v = 4k,\ 4k+1,$
- $v = 3k+1,\ 5k \quad$ if $k \equiv 1 \pmod 4$,
- $v = 3k,\ 5k+1 \quad$ if $k \equiv 3 \pmod 4$,

to have a k-sun system of K_v for every admissible v.

It is sufficient to construct...

Let k be a prime.

It is sufficient to construct a $2k$-sun system of K_v for

- $v = 8k,\ 8k+1$,
- $v = 7k+1,\ 9k \quad$ if $k \equiv 1 \pmod 8$,
- $v = 5k+1,\ 11k \quad$ if $k \equiv 3 \pmod 8$,
- $v = 5k,\ 11k+1 \quad$ if $k \equiv 5 \pmod 8$,
- $v = 7k,\ 9k+1 \quad$ if $k \equiv 7 \pmod 8$,

to have a $2k$-sun system of K_v for every admissible v.

Why is it sufficient to consider v in the interval $2k - 6k$?

Idea of the proof when k is even

For every $k \geq 4$ even we prove that:

$$\exists \text{ a } k\text{-sun system of } K_v$$
$$\Downarrow$$
$$\exists \text{ a } k\text{-sun system of } K_{v+4k}$$

then we get the existence of a k-sun system of K_{v+4kg} by induction on g.

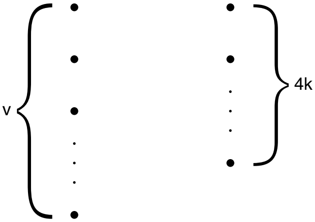
Idea of the proof when k is even

Case 1: v even $\Rightarrow v \equiv 0 \pmod 4$.

If there exist:

- a k-sun system of K_v
- a k-sun system of K_{4k}
- a k-sun system of $K_{v,4k}$

then there exists a k-sun system of K_{v+4k}.



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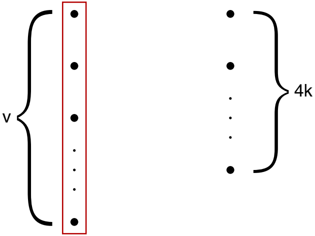

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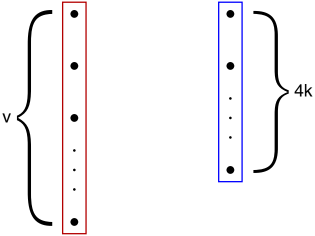

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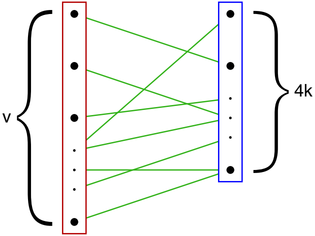

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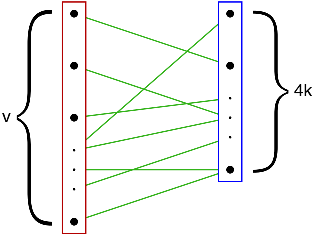

Case 2: v odd $\Rightarrow v \equiv 1 \pmod 4$.

If there exist:

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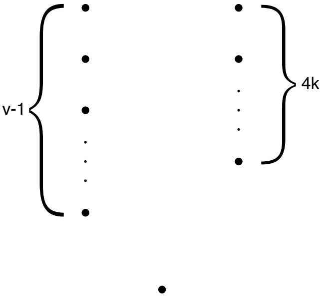

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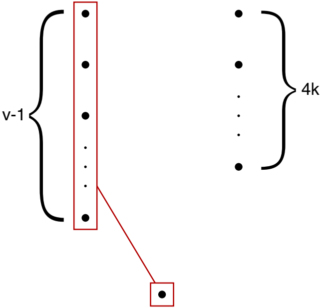

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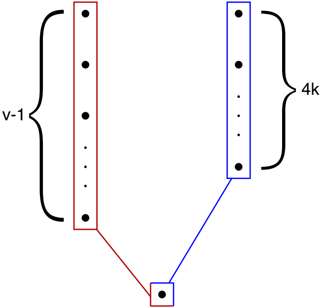

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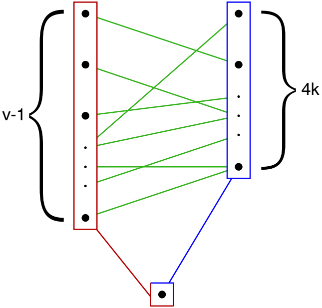

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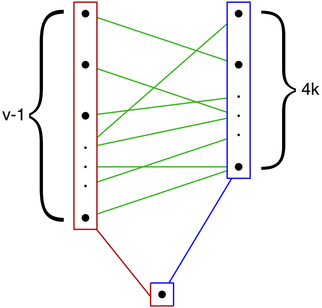

To conclude (k even)

$\exists$ a k-sun system of K_v

$\Downarrow$

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By induction on g we get:

Let $2k < v < 6k$. $\exists$ a k-sun system of K_v
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$\exists$ a k-sun system of K_{v+4kg} for every $g \geq 1$

$K_{4k} + v$

$$K_{4k} + v = K_{4k+v} \setminus K_v$$

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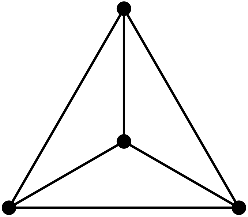

Figure: K_4

$K_{4k} + v$

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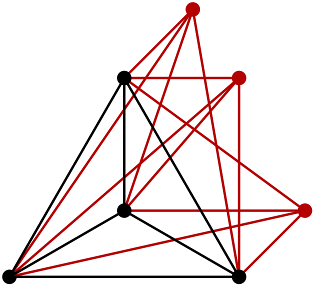

Figure: $K_4 + 3 = K_7 \setminus K_3$

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If there exist:

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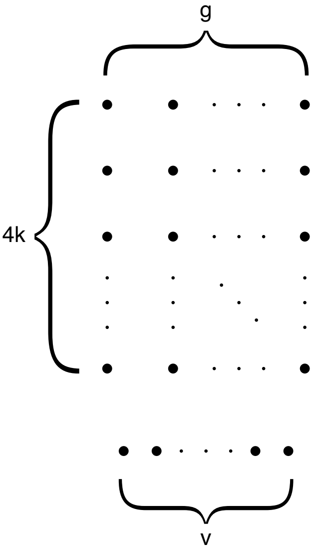

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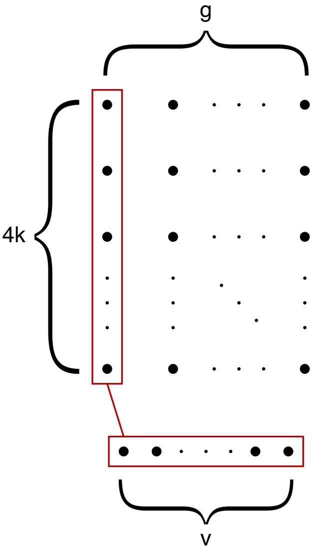

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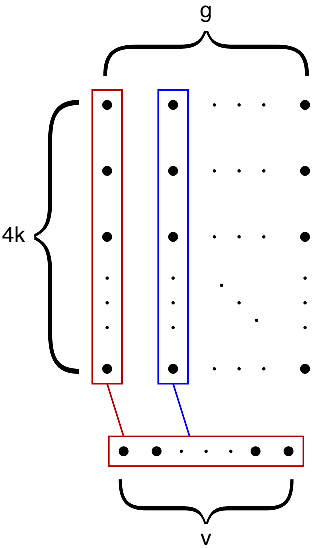

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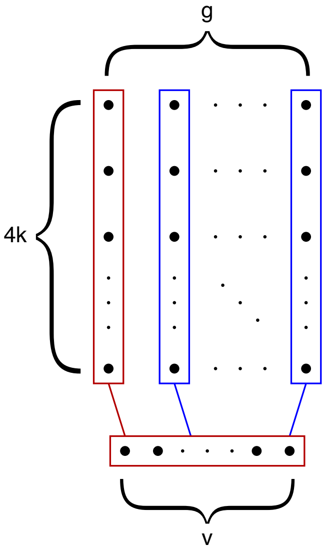

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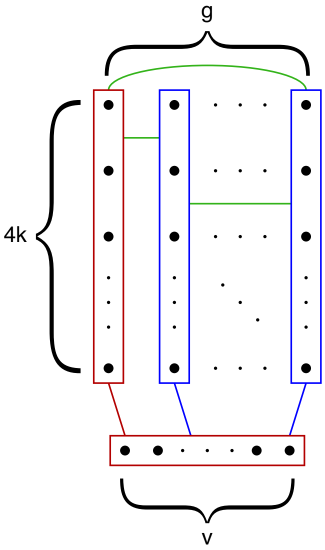

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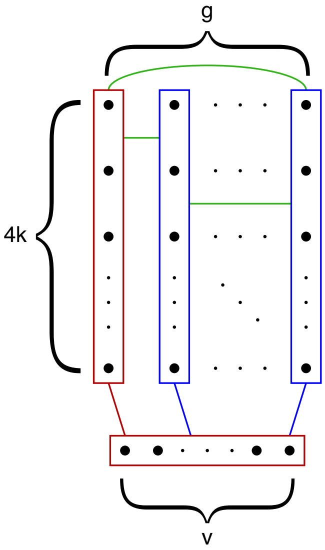
Case $v + 8k$: can be done in a similar way.



Sun system