Erdős-Rademacher Problem

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Some definitions

- Turán graph $T_r(n)$: balanced r-partite order-n
- $t_r(n) = e(T_r(n))$
- Turán'41: Max size of K_{r+1}-free order-n graph is $t_r(n)$
- Rademacher'41: $e(G_n) \geq t_2(n) + 1 \;\Rightarrow\; \geq \lfloor \frac{n}{2} \rfloor$ K_3's
- Erdős-Rademacher Problem:

$$g_r(n, m) = \min\{\#K_r(G_n) : e(G_n) = m\}$$

- Maximising $\#K_r$ is easy (Kruskal'63, Katona'68)

g_3 just above $t_2(n)$

- When is $g_3(n, t_2(n) + q) = q \cdot \lfloor \frac{n}{2} \rfloor$?
  - Erdős'55: $q \leq 3$
  - Erdős'62: $q \leq \varepsilon n$
  - Erdős'55: True for $q < n/2$?
    - $K_{k,k} + q$ edges versus $K_{k+1,k-1} + (q+1)$ edges
    - qk versus $(q+1)(k-1)$ triangles
  - Lovász-Simonovits'75: $q < n/2$
- Arbitrary $r \geq 3$:
  - Lovász-Simonovits'75,83:
    Solved g_r-problem for $t_k(n) \leq m \leq t_k(n) + \varepsilon_{k,r} n^2$

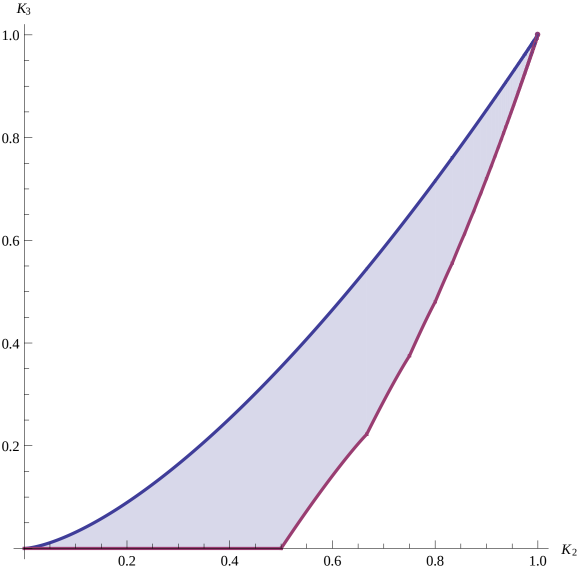
General m

- Conjecture (Lovász-Simonovits'75): If $n \geq n_0(r)$ then $g_r(n, m)$ is attained by a graph in $\mathcal{H}$
- $\mathcal{H} := \{$complete partite $+$ K_3-free inside a part$\}$

Asymptotic Version

- $g_r(a) := \lim_{n\to\infty} \frac{g_r\left(n, a\binom{n}{2}\right)}{\binom{n}{r}}$
- Conjecture + calculations
  - $g_r(a) = K_r$-density in $K_{cn,\ldots,cn,c'n}$ with $c \geq c'$
- Partial results: Goodman'59, Moon-Moser'62, Nordhaus-Stewart'63, Bollobás'76, Lovász-Simonovits'83
- Fisher'89: Determined $g_3(a)$ for $\frac{1}{2} \leq a \leq \frac{2}{3}$
- Razborov'08: Determined $g_3(a)$ for all a
- Nikiforov'11: Determined $g_4(a)$ for all a
- Reiher'16: Determined $g_r(a)$ for all r and a

Possible Edge-Triangle Densities in Limit



Stability

- Stability: Are all graphs with $m > t_{r-1}(n)$ edges and $g_r(n,m) + o(n^r)$ r-cliques $o(n^2)$-close to each other ?
- Graph family $\mathcal{H}' \subseteq \mathcal{H}$:
  - Take $K(A_2, \ldots, A_{k+1})$, $|A_1| = \ldots = |A_k| \geq |A_{k+1}|$
  - Move edges inside $A_k \cup A_{k+1}$ keeping it K_3-free
- No stability
- P.-Razborov'17: $\forall$ almost g_3-extremal G_n is $o(n^2)$-close in the edit distance to $\mathcal{H}'$
- Kim-Liu-P.-Sharifzadeh'20: $\forall$ $r \geq 4$ $\forall$ almost g_r-extremal G_n is $o(n^2)$-close to $\mathcal{H}' \cup \{K_r\text{-free}\}$

Exact Values of $g_r(n, m)$

- Previously known exact values for $n \to \infty$:
  - Goodman'59, Nordhaus-Stewart'63: $m = t_k(n)$
  - Lovász-Simonovits'83: $t_k(n) \leq m \leq t_k(n) + \varepsilon_k n^2$
  - Asymptotic result $\Rightarrow$ $m = e(K_{a,\ldots,a,b})$ for $a \geq b$
- Liu-P.-Staden'20: $r = 3$ & $m < \binom{n}{2} - o(n^2)$
  - at least one $g_3(n, m)$-extremal graph is complete partite + K_3-free
  - $\{g_3\text{-extremal graphs}\} = \mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2$

Asymptotic Structure for K_r-Minimisation

- Kim-Liu-P.-Sharifzadeh'20: $\forall\ r \geq 4\ \forall$ almost g_r-extremal G_n is $o(n^2)$-close to $\mathcal{H}' \cup \{K_r\text{-free}\}$
- $\mathcal{H}' = \{K_{cn,\dots,cn,c'n}$ with the last two parts modified$\}$

Graphons

- Graphon: Measurable function $W : [0,1]^2 \to [0,1]$ st $W(x,y) = W(y,x)$
- $G \mapsto W^G$, the adjacency function of G
- (Homomorphism) density of $F = ([k], E)$ in G:

$$t(F,G) := \frac{1}{|V(G)|^k} \#\{f : [k] \to V(G) \text{ st } f(E) \subseteq E(G)\}$$

- (Homomorphism) density of $F = ([k], E)$ in W:

$$t(F,W) := \int_{[0,1]^k} \prod_{ij \in E} W(x_i, x_j) \; \mathrm{d}x_1 \ldots \mathrm{d}x_k$$

- $t(F,G) = t(F, W^G)$
- G_n converge to W if $\forall\, F \;\; t(F, G_n) \to t(F, W)$

Limit Version

- $\mathcal{W} := \{\text{graphons}/\sim\}$ is compact
- $t(H, \cdot) : \mathcal{W} \to [0, 1]$ is continuous
- $\Rightarrow\ g_r(a) = \min\{t(K_r, W) : t(K_2, W) = a\}$
- W is g_r-extremal if $t(K_r, W) = g_r(t(K_2, W))$
- $\mathcal{H}' = \{K_{cn,\dots,cn,c'n}$ with the last two parts modified$\}$
- $\mathrm{LIM}(\mathcal{H}') := \{$limits of convergent sequences from $\mathcal{H}'\}$
- Kim-Liu-P.-Sharifzadeh'20: $\forall\ r \geq 4$
  $\{g_r\text{-extremal graphons}\} = \mathrm{LIM}(\mathcal{H}') \cup \{W : t(K_r, W) = 0\}$
- Implies the discrete theorem

Starting proof

- Fix any g_r-extremal W with $t(K_r, W) > 0$.
- $a := t(K_2, W)$
- $k \in \mathbb{N}$: $1 - \frac{1}{k} \leq a < 1 - \frac{1}{k+1}$
- $c \in (\frac{1}{k+1}, \frac{1}{k}]$: $t(K_2, K_{c,\ldots,c,(1-kc)}) = a$
- P.-Razborov'17: $t(K_3, W) = g_3(a) \;\Rightarrow\; W \in \mathrm{LIM}(\mathcal{H}')$
- Done unless $t(K_3, W) > g_3(a)$
- Lovász-Simonovits'83:
  $a = 1 - \frac{1}{k} \;\Rightarrow\; W = k$-partite Turan graphon
- $\Rightarrow\; a \neq 1 - \frac{1}{k}$

Razborov's Differential Calculus I

- g_r is differentiable at a
- $g_r'(a) = (k-1)^{(r-2)} c^{r-2}$
- $t_{x_1,x_2}(K_t^-, W) := \int_{[0,1]^{r-2}} \prod_{ij \in \binom{[t]}{2} \setminus \{12\}} W(x_j, x_j) \, \mathrm{d}x_3 \ldots \mathrm{d}x_t$
- $t_{xy}(K_3^-, W) := \int W(x,z) W(y,z) \, \mathrm{d}z$
- $t_{xy}(K_3^-, G_n) := \frac{1}{n} \#\{$2-paths between $x, y \in V\}$
- Razborov'07: For r-extremal W and a.e. $xy \in [0,1]^2$ $\big(W(x,y) > 0 \;\Rightarrow\; t_{xy}(K_r^-, W) \le g_r'(a)\big)$

Razborov's Differential Calculus II

- $t_{x_1}(K_t, W) := \int_{[0,1]^{t-1}} \prod_{ij \in \binom{[t]}{2}} W(x_i, x_j)\, \mathrm{d}x_2 \ldots \mathrm{d}x_t$
- Degree $d(x) := \int_0^1 W(x, y)\, \mathrm{d}y = t_x(K_2, W)$
- Add measure-ε copies of x. Changes:
  - $\partial t(K_2, W) = \frac{2\varepsilon d(x)+a}{(1+\varepsilon)^2} - a = 2(d(x) - a)\varepsilon + O(\varepsilon^2)$
  - $\partial t(K_r, W) = r(t_x(K_r, W) - g_r(a))\varepsilon + O(\varepsilon^2)$
    $\geq \; \partial g_r \; = \; g_r'(a) \cdot 2(d(x) - a)\varepsilon + O(\varepsilon^2)$
- $\Rightarrow \; t_x(K_r, W) - g_r(a) \geq \frac{1}{r}\, g_r'(a) \cdot 2(d(x) - a)$
- "Remove" $x \; \Rightarrow \; \leq$ holds a.e.
- $\Rightarrow \; f_r(x) := C_1 d(x) + C_2 - t_x(K_r, W)$ is 0 a.e.

Properties of f_t

- Arbitrary $t \in \mathbb{N}$
- $f_t(x) := C_{1,t} d(x) + C_{2,t} - t_x(K_t, W)$
  - $C_{1,t} = (t-1)(k-1)^{(t-2)} c^{t-2}$
  - $C_{2,t} = -(t-2)(k-1)(k-1)^{(t-2)} c^{t-1}$
- Lemma: $\int f_t(x)\,\mathrm{d}x = g_t(a) - t(K_t, W)$, any $t \geq 3$
- $t(K_3, W) > g_3(a) \;\Rightarrow\; f_3 < 0$ on positive measure
- Pick x with $f_3(x) < 0$, $d(x) > 0$ and $f_r(x) = 0$
- W': graphon “induced” by neighbourhood of x in W
- $t(K_{t-1}, W') = t_x(K_t, W)/d^{t-1}(x)$
- $\Rightarrow$ W' has too large edge density given K_{r-1}-density

Concluding Remarks

- Open: exact result for $r \geq 4$
- Open: exact result for $r = 3$ and remaining m
- Conjecture (Lovász-Simonovits'75): If $n \geq n_0(r)$ then $g_r(n, m)$ is attained by a graph in $\mathcal{H}$
- $\mathcal{H} := \{\text{complete partite} + K_3\text{-free inside a part}\}$
- Liu-P.-Staden'20: One counterexample to $g_3(n, m) = h_3(n, m) \quad \Rightarrow \quad$ false for every large n

Thank you!