Contact CR-submanifolds in odd-dimensional spheres: new examples

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din IAŞI

Table of Contents

CR-submanifolds

History

$(M, g) \underset{iso}{\hookrightarrow} (\widetilde{M}, \widetilde{g}, J)$ – Kähler manifold

$T(M)$ its tangent bundle; $T(M)^{\perp}$ its normal bundle

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Two important situations occur:

- **$T_x(M)$ is invariant under the action of J:**

$$J(T_x(M)) = T_x(M) \text{ for all } x \in M$$

M is called *complex* submanifold or holomorphic submanifold

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$(M, g) \underset{iso}{\hookrightarrow} (\widetilde{M}, \widetilde{g}, J)$ – Kähler manifold

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Two important situations occur:

- **$T_x(M)$ is anti-invariant under the action of J:**

$$J(T_x(M)) \subset T(M)_x^{\perp} \text{ for all } x \in M$$

M is know as a totally real submanifold

History

In **1978** A. Bejancu

- *CR-submanifolds of a Kähler manifold. I,*
  Proc. Amer. Math. Soc., 69 (1978), 135-142
- *CR- submanifolds of a Kähler manifold. II,*
  Trans. Amer. Math. Soc., 250 (1979), 333-345

started a study of the geometry of a class of submanifolds situated between the two classes mentioned above.

Such submanifolds were named *CR–submanifolds*:

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Such submanifolds were named *CR−submanifolds*:

M is a ***CR-submanifold*** *of a Kähler manifold $(\widetilde{M}, \widetilde{g}, J)$ if there exists a holomorphic distribution $\mathcal{D}$ on M, i.e. $J\mathcal{D}_x = \mathcal{D}_x$, $\forall x \in M$ and such that its orthogonal complement $\mathcal{D}^{\perp}$ is anti-invariant, namely $J\mathcal{D}_x^{\perp} \subset T(M)_x^{\perp}$, $\forall x \in M$.*

Sasakian manifolds

$(\widetilde{M}^{2m+1}, \phi, \xi, \eta, \widetilde{g})$ Sasakian manifold:

$\phi \in \mathcal{T}_1^1(\widetilde{M})$, $\xi \in \chi(\widetilde{M})$, $\eta \in \Lambda^1(\widetilde{M})$:

$$\phi^2 = -I + \eta \otimes \xi,\ \phi\xi = 0,\ \eta \circ \phi = 0,\ \eta(\xi) = 1$$

$d\eta(X, Y) = \widetilde{g}(\phi X, Y)$ (the contact condition)

$\widetilde{g}(\phi X, \phi Y) = \widetilde{g}(X, Y) - \eta(X)\eta(Y)$ (the compatibility condition)

$N = N_\phi + 2d\eta \otimes \xi = 0$ (the normality condition)

$$(\widetilde{\nabla}_U \phi)V = -\widetilde{g}(U, V)\xi + \eta(V)U, \quad U, V \in \chi(\widetilde{M})$$

Semi-invariant submanifolds in almost contact metric manifolds

Contact CR-submanifolds

Sasakian geometry: odd dimensional version of Kählerian geometry,
A new concept: *contact CR-submanifold*:

Contact CR-submanifolds

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A new concept: *contact CR-submanifold*:
a submanifold M of an almost contact Riemannian manifold $(\widetilde{M}, (\phi, \xi, \widetilde{\eta}, \widetilde{g}))$ carrying an invariant distribution $\mathcal{D}$, i.e. $\phi_x \mathcal{D}_x \subseteq \mathcal{D}_x$, for any $x \in M$, such that the orthogonal complement $\mathcal{D}^\perp$ of $\mathcal{D}$ in $T(M)$ is anti-invariant, i.e. $\phi_x \mathcal{D}_x^\perp \subseteq T(M)_x^\perp$, for any $x \in M$.

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This notion was introduced by A.Bejancu & N.Papaghiuc in

Semi-invariant submanifolds of a Sasakian manifold,
An. Şt. Univ. "Al.I.Cuza" Iaşi, Matem., 1(1981), 163-170.

by using the terminology of *semi-invariant submanifold*.

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by using the terminology of *semi-invariant submanifold*.
It is customary to require that ξ be tangent to M rather than normal which is too restrictive (K. Yano & M. Kon): M must be anti-invariant.

Contact CR-submanifolds

Given a contact CR submanifold M of a Sasakian manifold $\widetilde{M}$
either $\xi \in \mathcal{D}$, or $\xi \in \mathcal{D}^{\perp}$. Therefore

$$T(M) = H(M) \oplus \mathbf{R}\xi \oplus E(M)$$

$H(M)$ is the maximally complex, distribution of M; $\phi E(M) \subseteq T(M)^{\perp}$.

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organize M as a contact CR submanifold

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$H(M)$ is never integrable (e.g. Capursi & Dragomir - 1990)

Some inequalities

Theorem (Papaghiuc - 1984, M. - 2005)

Let $\widetilde{M}^{2m+1}(c)$ be a Sasakian space form and let $M = N^{\top} \times N^{\perp}$ be a contact CR product in $\widetilde{M}$. Then the norm of the second fundamental form of M satisfies the inequality

$$||h||^2 \geq q \ \left((c+3)s+2\right).$$

"=" holds if and only if both $N^{\top}$ and $N^{\perp}$ are totally geodesic in $\widetilde{M}$.

Some inequalities

$$r : S^{2s+1} \times S^{2q+1} \longrightarrow S^{2m+1} \qquad \mathbf{m = sq + s + q}$$

$$(x_0,\ y_0, \ldots, x_s,\ y_s;\ u_0,\ v_0, \ldots, u_q,\ v_q) \longmapsto (\ldots, x_j u_\alpha - y_j v_\alpha,\ x_j v_\alpha + y_j u_\alpha, \ldots)$$

$M = S^{2s+1} \times S^p \longrightarrow S^{2s+1} \times S^{2q+1} \xrightarrow{r} S^{2m+1}$

contact CR product in S^{2m+1} for which the equality holds.

Some inequalities

Theorem (Papaghiuc - 1984, M. - 2005)

Let M be a strictly proper contact CR product in a Sasakian space form $\widetilde{M}^{2m+1}(c)$, with $c \neq -3$. Then

$$m \geq sq + s + q.$$

Some inequalities

Theorem (M. - 2005, Theorem 3.3)

Let $M = N_1 \times_f N_2$ be a contact CR warped product of a Sasakian space form $\widetilde{M}^{2m+1}(c)$, that is a contact CR-submanifold in $\widetilde{M}$, such that N_1 is φ-invariant and tangent to ξ, while N_2 is φ-anti-invariant. Then the second fundamental form of M satisfies the following inequality

$$||h||^2 \geq 2p \left[||\nabla \ln f||^2 - \Delta \ln f + \frac{c+3}{2} s + 1 \right].$$

Here f is the warping function which has to satisfy $\xi(f) = 0$ and Δf is the Laplacian of f.

Interesting result in S^7

Theorem (M. - 2005)

Let $M = N^T \times N^\perp$ be a strictly proper contact CR product in S^7 whose second fundamental form has the norm $\sqrt{6}$. Then M is the Riemannian product between S^3 and S^1 and, up to a rigid transformation of $\mathbf{R}^8$ the embedding is given by

$$r : S^3 \times S^1 \longrightarrow S^7$$
$$r(x_1,\, y_1,\, x_2,\, y_2,\, u,\, v) = (x_1 u,\, y_1 u,\, -y_1 v,\, x_1 v,\, x_2 u,\, y_2 u,\, -y_2 v,\, x_2 v).$$

Minimal contact CR submanifolds in $\mathbb{S}^{2n+1}$ satisfying the $\delta(2)$-Chen's equality

$\delta(2)$ invariant

In 1993, B.-Y. Chen:

$$\delta(2)(p) = \tau(p) - (\inf K)(p)$$

where

$$(\inf K)(p) = \inf \{K(\pi) \mid \pi \text{ is a 2-dimensional subspace of } T_pM\}$$

and $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$ denotes the scalar curvature

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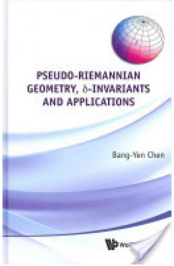

$\delta(2)$ invariant; $\delta(2)$ ideals

In real space forms $\widetilde{M}(c)$:

$$\delta(2) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}(n-2)(n+1)c.$$

Note that, for $n = 2$, both sides of the above inequality are zero.

A submanifold is called $\delta(2)$ ideal if and only if at each point it realizes the equality in the above inequality.

Recent results in odd dimensional spheres

M.I. Munteanu, L. Vrancken:
Minimal contact CR submanifolds in $\mathbb{S}^{2n+1}$ satisfying the $\delta(2)$-Chen's equality,
Journal of Geometry and Physics, **75** (2014), 92–97.

Let M be a minimal proper contact CR submanifold of dimension n in the $(2m+1)$-dimensional sphere $\mathbb{S}^{2m+1}$.
Assuming that the contact CR structure is proper:

- $\mathbf{q = 1}$.
- n **is an even number**.

Recent results in odd dimensional spheres

Theorem (M., Vrancken - 2014)

Let M^n be a proper, minimal, contact CR-submanifold of $\mathbb{S}^{2m+1}$ which is a $\delta(2)$-ideal. Then n is even and there exists a totally geodesic Sasakian $\mathbb{S}^{n+1}$ in $\mathbb{S}^{2m+1}$ containing M as a hypersurface.

Recent results in odd dimensional spheres

Theorem (M., Vrancken - 2014)

Let M^n, n even, be a proper, minimal, contact CR-submanifold of $\mathbb{S}^{2m+1}$ which is a $\delta(2)$-ideal. Then there exists a minimal surface in $\mathbb{S}^{n+1}$ such that locally M can be considered as the unit normal bundle of this minimal surface.

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Theorem (M., Vrancken - 2014)

Let M^n, n even, be a proper, minimal, contact CR-submanifold of $\mathbb{S}^{2m+1}$ which is a $\delta(2)$-ideal. Then M can be locally considered as the unit normal bundle of the Clifford torus $S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3 \subset \mathbb{S}^{n+1}$.

Contact CR submanifolds in $\mathbb{S}^7$

Four dimensional minimal contact CR submanifolds in $\mathbb{S}^7$ satisfying Chen's equality

We find (locally) coordinates on M such that the immersion $F : M \longrightarrow \mathbb{S}^7$ is expressed as

$$\begin{aligned} F(s,t,u,v) &= \cos s \cos t \cos u \; e_1 + \sin s \cos t \cos u \; e_2 \\ &\quad + \cos s \cos t \sin u \; e_3 + \sin s \cos t \sin u \; e_4 \\ &\quad + \sin t \cos v \; e_5 + \sin t \sin v \; e_6. \end{aligned}$$

Remark that M lies in a 5-dimensional sphere in a 6-dimensional J invariant subspace of $\mathbb{R}^8$: $e_2 = Je_1$, $e_4 = Je_3$, $e_6 = Je_5$.

New problem:

Let M be a contact CR submanifold in the Sasakian sphere $\mathbb{S}^7$.
Recall that:

$$T(M) = H(M) \oplus E(M) \oplus [\xi]$$
$$T^{\perp}(M) = \varphi E(M) \oplus \nu(M)$$

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Let M be a contact CR submanifold in the Sasakian sphere $\mathbb{S}^7$.
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$$T^{\perp}(M) = \varphi E(M) \oplus \nu(M)$$

Problem. Find all proper contact CR submanifolds in $\mathbb{S}^7$ such that

$$h(H(M), H(M)) = 0 \quad \& \quad h(E(M), E(M)) = 0.$$

Work in progress

We have:

$$\mathbf{s+q+r=3}$$

where $2s = \dim(H(M))$, $q = \dim(E(M))$, $2r = \dim(\nu(M))$.
Then:

I. $s = q = r = 1$, hence $\dim(M) = 4$

II. $s = 1$, $q = 2$, $r = 0$ hence $\dim(M) = 5$

III. $s = 2$, $q = 1$, $r = 0$ hence M is a hypersurface in $\mathbb{S}^7$

4-dimensional contact CR submanifolds in $\mathbb{S}^7$

Problem solved: with M. Djorić and L. Vrancken;
Math. Nachr. 290 (2017) 16, 2585–2596.

4-dimensional contact CR submanifolds in $\mathbb{S}^7$

Theorem (Djorić, M., Vrancken, 2017)

Let M be a 4-dimensional nearly totally geodesic contact CR-submanifold in $\mathbb{S}^7$. The M is locally congruent with one of the following immersions:

① $$F(u,v,s,t) = \Big(\cos s \sin t \; \mathrm{e}^{i\lambda u}, \cos t \sin v \; \mathrm{e}^{i\mu u}, -\sin s \sin t \; \mathrm{e}^{i\lambda u}, \cos t \cos v \; \mathrm{e}^{i\mu u}\Big)$$

② $F : \mathbb{S}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^8$, $F(y,t) = (\cos t \; y, \sin t \; y)$.

③ $$F(v,u,t,s) = e^{iv}\big(e^{is}\cos t \cos t_0 \cos u + e^{-is}\sin t \sin t_0, -ie^{is}\cos t \sin t_0 \cos u + ie^{-is}\sin t \cos t_0, e^{is}\cos t \sin u, 0\big).$$

5-dimensional contact CR submanifolds in $\mathbb{S}^7$

Problem solved: with M. Djoric – 2020

5-dimensional contact CR submanifolds in $\mathbb{S}^7$

Theorem (Djoric and M., 2020)

Let M be a five-dimensional proper nearly totally geodesic contact $CR-$submanifold of seven-dimensional unit sphere. Then M is locally congruent to warped products $\mathbb{S}^3 \times_{f_1} \mathbb{S}^1 \times_{f_2} \mathbb{S}^1$ and $\mathbb{S}^3 \times_f \mathbb{S}^2$ via the immersions

①

$$F(x,y,z,u,v) = \Big(\cos y \cos u \cos(z+\tfrac{x}{2}), \cos y \cos u \sin(z+\tfrac{x}{2}), \\ \sin y \sin v \cos(z-\tfrac{x}{2}), \sin y \sin u \sin(z-\tfrac{x}{2}), \\ \sin y \cos v \cos(z-\tfrac{x}{2}), \sin y \cos v \sin(z-\tfrac{x}{2}), \\ \cos y \sin u \cos(z+\tfrac{x}{2}), \cos y \sin u \sin(z+\tfrac{x}{2}) \Big)$$

for $y \in (0, \frac{\pi}{2}), x, z, u, v \in \mathbb{R}$.

5-dimensional contact CR submanifolds in $\mathbb{S}^7$

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②
$$\begin{aligned} F(\dots) = & -\cos y(1-T)\left(\cos\left(z+\tfrac{x}{2}\right), \sin\left(z+\tfrac{x}{2}\right), 0,0,0,0,0,0\right) \\ & +\cos yTu\left(0,0,\cos\left(z+\tfrac{x}{2}\right), \sin\left(z+\tfrac{x}{2}\right), 0,0,0,0,\right) \\ & +\cos yTv\left(0,0,0,0,0,0,\cos\left(z+\tfrac{x}{2}\right), \sin\left(z+\tfrac{x}{2}\right)\right) \\ & +\sin y\left(0,0,0,0,\cos\left(z-\tfrac{x}{2}\right), \sin\left(z-\tfrac{x}{2}\right), 0,0\right). \end{aligned}$$

where $T = 2/(1+u^2+v^2)$.

4-dimensional contact CR submanifolds in $\mathbb{S}^7$

Example 1.

$$\tilde{F} : \mathbb{R}^4 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^8$$
$$\tilde{F}(x_1, y_1, x_2, y_2; u, v) = (x_1 u, y_1 u, x_2 u, y_2 u, x_1 v, y_1 v, x_2 v, y_2 v),$$

is not an immersion.

4-dimensional contact CR submanifolds in $\mathbb{S}^7$

Example 1.

Its restriction $F : M = \mathbb{S}^3 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^7$

$F(x_1, y_1, x_2, y_2; u, v) = (x_1 u, y_1 u, x_2 u, y_2 u, x_1 v, y_1 v, x_2 v, y_2 v)$

(with standard metrics) is an isometric immersion.

4-dimensional contact CR submanifolds in $\mathbb{S}^7$

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(with standard metrics) is an isometric immersion.
the characteristic vector field
$\xi = (-y_1, x_1, -y_2, x_2) = Jp$ for $p = (x_1, y_1, x_2, y_2)$ of $\mathbb{S}^3$

$$F_* \xi = \xi$$

$$\zeta = (-x_2, y_2, x_1, -y_1)\ , \ \mu = (-y_2, -x_2, y_1, x_1) \text{ and } \psi = (-v, u)$$

Set $H(M) = \mathrm{span}\{\zeta, \mu\}$ and $E(M) = \mathrm{span}\{\psi\}$
M becomes a **nearly t.g.** contact CR-submanifold of $\mathbb{S}^7$

4-dimensional contact CR submanifolds in $\mathbb{S}^7$

Example 1.

Its restriction $F : M = \mathbb{S}^3 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^7$ Chen type inequality

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4-dimensional contact CR submanifolds in $\mathbb{S}^7$

Example 2.

$$F : M = \mathbb{S}^3 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^7$$
$$F(x_1, y_1, x_2, y_2; u, v) = (x_1 u, y_1 u, x_1 v, y_1 v, x_2, y_2, 0, 0),$$

with the warped metric $g_M = g_{\mathbb{S}^3} + f^2 g_{\mathbb{S}^2}$ where
$f : D \subset \mathbb{S}^3 \to \mathbb{R}, \quad f(x_1, y_1, x_2, y_2) = \sqrt{x_1^2 + y_1^2}$
F is an isometric immersion and we have:

4-dimensional contact CR submanifolds in $\mathbb{S}^7$

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F is an isometric immersion and we have:

(i) M is **nearly totally geodesic**;

(ii) M is minimal and satisfies the equality in two Chen type inequalities from my thesis;

(iii) M is a $\delta(2)$-ideal in $\mathbb{S}^7$. [M. and Vrancken - 2014]

5-dimensional contact CR submanifolds in $\mathbb{S}^7$

Example 1. [Djoric and M., 2020]

$$F : M = \mathbb{S}^3 \times \mathbb{S}^2 \longrightarrow \mathbb{S}^7$$
$$F(x_1, y_1, x_2, y_2; u, v, w) = (x_1 u, y_1 u, x_1 v, y_1 v, x_1 w, y_1 w, x_2, y_2).$$

isometric immersion: warped metric on M

$g_M = g_{\mathbb{S}^3} + f^2 g_{\mathbb{S}^2}$, where $f : D \subset \mathbb{S}^3 \to \mathbb{R}$, $\quad f(x_1, y_1, x_2, y_2) = \sqrt{x_1^2 + y_1^2}$.

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Proposition.

(i) M is **nearly totally geodesic**;

(ii) M is minimal and satisfies the equality in two Chen type inequalities from my thesis.

5-dimensional contact CR submanifolds in $\mathbb{S}^7$

Example 2. [Djoric and M., 2020]

$$F : \mathbb{S}^3 \times_{f_1} \mathbb{S}^1 \times_{f_2} \mathbb{S}^1 \longrightarrow \mathbb{S}^7$$

$$F(\mathbf{x_1}, \mathbf{y_1}, \mathbf{x_2}, \mathbf{y_2}; \mathbf{u}, \mathbf{v}; \mathbf{a}, \mathbf{b}) = (\mathbf{ux_1}, \mathbf{uy_1}, \mathbf{vx_1}, \mathbf{vy_1}, \mathbf{ax_2}, \mathbf{ay_2}, \mathbf{bx_2}, \mathbf{by_2}),$$

where the warping functions $f_1, f_2 : D \subset \mathbb{S}^3 \to (0, \infty)$ are given by

$$f_1(\mathbf{x_1}, \mathbf{y_1}, \mathbf{x_2}, \mathbf{y_2}) = \sqrt{\mathbf{x}_1^2 + \mathbf{y}_1^2}$$

$$f_2(\mathbf{x_1}, \mathbf{y_1}, \mathbf{x_2}, \mathbf{y_2}) = \sqrt{\mathbf{x}_2^2 + \mathbf{y}_2^2}.$$

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