Analytic properties of the complete formal normal form for the Bogdanov–Takens singularity

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Introduction

Recall that a singularity of a planar vector field is **elementary** if at least one eigenvalue of the linearization matrix is nonzero. We can write

$$\boldsymbol{V} = \boldsymbol{V}_0 + \cdots = \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y} + \ldots,$$

where $\lambda_1 \neq 0$. The normal form and its analytic properties depend on the ratio

$$\lambda := \lambda_2 / \lambda_1$$

of the eigenvalues.

In the **focus case**, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the formal normal form (which is linear) is analytic.

In the **node case**, $\lambda \in \mathbb{R}_{>0}$, the normal form (which is polynomial) is also analytic.

In the **non-resonant saddle case**, $\lambda \in \mathbb{R}_{<0} \setminus \mathbb{Q}$, the (linear) normal form can be analytic or non-analytic, depending on the approximation properties of the number λ by rationals; but for dense set of λ's one can show that for general nonlinear terms the normal form is non-analytic.

In the **resonant saddle case**, $\lambda \in \mathbb{Q}_{\leq 0}$ (we include into this class also the saddle-node case $\lambda = 0$), the formal normal form is generally non-analytic.

In the case of the **Bogdanov–Takens singularity** , i.e., with nonzero nilpotent linear part, the complete formal normal form (orbital and non-orbital) was obtained only recently. Using the Takens result we can assume that we deal with vector fields of the form

$$\boldsymbol{V}_H + \boldsymbol{W},$$

where

$$\boldsymbol{V}_H = (y + (\lambda + 1)\, x^r)\, \frac{\partial}{\partial x} - \lambda r x^{2r-1} \frac{\partial}{\partial y}$$

is a quasi-homogeneous vector field with respect to the grading $\deg_H$ such that

$$\deg_H x = 1, \quad \deg_H y = r,$$

and $\boldsymbol{W}$ contains higher degree terms. Above $r \in \frac{1}{2}\mathbb{Z}$ and $\lambda = -1$ when $r \notin \mathbb{Z}$ (we have the so-called generalized cusp).

The relation with elementary singularities follows from the fact that, after putting $z = x^r$ and dividing by rx^{r-1}, one arrives at the linear vector field

$$(y + (\lambda + 1)\, z)\, \frac{\partial}{\partial z} - \lambda z \frac{\partial}{\partial y},$$

such that λ is the ratio of its eigenvalues.

The BT singularities were divided into three types. **Type I** includes the cases with $\lambda \notin \mathbb{Q}$ (nonresonant) and $\lambda = \frac{k}{l} \in \mathbb{Q}_{>0}$, $k, l > 1$, gcd$(k, l) = 1$ (analogues to the $k : l$ resonant nodes). **Type II** includes analogues to resonant nodes with $l = 1$ ($\lambda = k > 0$) and **Type III** includes the cases corresponding to the $k : -l$ resonant saddles ($\lambda = -\frac{k}{l} \in \mathbb{Q}_{<0}$, gcd$(k, l) = 1$ (including $\lambda = 0$).

The general **normal form** looks as follows:

$$\Psi(x) \left\{ \boldsymbol{V}_H + \Phi(x) \boldsymbol{E}_H \right\},$$

where

$$\boldsymbol{E}_H = x \frac{\partial}{\partial x} + ry \frac{\partial}{\partial y}$$

is the **quasi-homogeneous Euler vector field** and $\Phi(x) = x^p \varphi(x) = \sum_{i \in \mathcal{I}(\Phi)} a_i x^i$, $\Psi(x) = 1 + x^q \psi(x) = \sum_{i \in \mathcal{I}(\Psi)} b_i x^i$ are formal power series with specified sets of powers $\mathcal{I}(\Phi)$ and $\mathcal{I}(\Psi)$.

Above $\boldsymbol{V}_H + \Phi(x)\boldsymbol{E}_H$ is the **orbital normal form** and $\Psi(x)$ is the **orbital factor**.

In particular, for Type I we have $\mathcal{I}(\Phi) = \mathbb{Z}_{\geq r} \setminus I_1$ and $\mathcal{I}(\Psi) = \mathbb{Z}_{\geq 0} \setminus I_1$, where

$$I_k = \{j : j + k = 0 \bmod r\}.$$

For other types the indices sets $\mathcal{I}(\Phi)$ and $\mathcal{I}(\Psi)$ are more complicated and for Type II the normal form is slightly different.

But sometimes this choice is not the best ones from the point of view of its analyticity; some estimates turn out too complicated. In those cases we choose other versions of the normal forms.

Theorem 1. *In the case $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the corresponding normal form is analytic.*

Theorem 2. *In the case $\lambda > 0$ the corresponding normal form is analytic.*

Theorem 3. *In the case* $\lambda \in \mathbb{R}_{<0} \diagdown \mathbb{Q}$, for λ *from a dense set and generic perturbation* $\boldsymbol{W}$, *the normal form is non-analytic.*

Theorem 4. *In the case of the* $k : -l$ *resonance and* $p < \infty$, *in* $\Phi = x^p \varphi(x)$, *the normal form is non-analytic in general and for* $p = \infty$ *it is analytic.*

Koszul complexes and homological operators

Let

$$\mathcal{F}_d = \{f \in \mathbb{C}[x,y] : \deg f = d\}\,,$$
$$\mathcal{Z}_d = \{\boldsymbol{Z} \in \mathcal{Z} : \deg \boldsymbol{Z} = d\}\,,$$

denote spaces of quasi-homogeneous polynomials and of quasi-homogeneous vector fields of degree d.

We consider vector fields of the form $\boldsymbol{V} = \boldsymbol{V}_0 + \ldots$, where $\boldsymbol{V}_0$ is as above in the case of elementary singularity, and $\boldsymbol{V}_0 = \boldsymbol{V}_H$ in the case of the BT singularity. Introduce the operators

$$\begin{array}{rcl} A(\boldsymbol{V})f & = & f \cdot \boldsymbol{V}, \\ B(\boldsymbol{V})\boldsymbol{Z} & = & \boldsymbol{V} \wedge \boldsymbol{Z}/\partial_x \wedge \partial_y, \\ C(\boldsymbol{V})f & = & \boldsymbol{V}(f), \\ D(\boldsymbol{V})f & = & \boldsymbol{V}(f) - \mathrm{div}(\boldsymbol{V})f. \end{array}$$

Consider the following diagram, with rows that form the so-called **Koszul complexes**:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}_{\geq d-r+1} & \xrightarrow{A(\boldsymbol{V})} & \mathcal{Z}_{\geq d} & \xrightarrow{B(\boldsymbol{V})} & \mathcal{F}_{\geq d+2r} & \longrightarrow & 0 \\
 & & \downarrow C(\boldsymbol{V}) & & \downarrow \mathrm{ad}_{\boldsymbol{V}} & & \downarrow D(\boldsymbol{V}) & & \\
0 & \longrightarrow & \mathcal{F}_{\geq d} & \xrightarrow{A(\boldsymbol{V})} & \mathcal{Z}_{\geq d+r-1} & \xrightarrow{B(\boldsymbol{V})} & \mathcal{F}_{\geq d+3r-1} & \longrightarrow & 0
\end{array}
$$

The operators $C(\boldsymbol{V})$, $\mathrm{ad}_{\boldsymbol{V}}$ and $D(\boldsymbol{V})$ are called the **homological operators**.

In the elementary case we have $r = 1$ and $\deg = \deg_H$.

The above diagram is commutative.

If $\lambda \neq 0$, i.e., when $x = y = 0$ is an isolated singularity, then the Koszul complexes are exact. If $\lambda = 0$ then the situation is only slightly more complicated.

$\ker C(\boldsymbol{V})$ consists of the **first integrals** (FIs) F of $\boldsymbol{V}$.

$\ker D(\boldsymbol{V})$ consists of the **inverse integrating multipliers** (IIMs) M for $\boldsymbol{V}$, i.e., $\mathrm{div} M^{-1}\boldsymbol{V} = 0$.

For the non-resonant singularities ($\lambda \notin \mathbb{Q}$ or Type I) we have:

$\ker C(\boldsymbol{V}_0) = 0$ and $\ker D(\boldsymbol{V}_0) = 0$.

For singularities of Type II (with $\lambda = k \in \mathbb{Z}_{>0}$) we have:

$\ker C(\boldsymbol{V}_0) = 0$ and $\ker D(\boldsymbol{V}_0) = \mathbb{C} \cdot x^{k+1}$ or $= \mathbb{C} \cdot (y + x^r)^{k+1}$.

For the $k : -l$ resonant singularities (Type III) we have:

$\ker C(\boldsymbol{V}_0) = \mathbb{C}[[F]]$, $F = x^k y^l$, or $F = (y + x^r)^k (y + \lambda x^r)^l$ and

$\ker D(\boldsymbol{V}_0) = xy \cdot \mathbb{C}[[F]]$ or $= (y + x^r)(y + \lambda x^r) \cdot \mathbb{C}[[F]]$.

Remark. In our previous work the images of the above homological operators are described explicitly in terms of periods of certain Schwarz–Christoffel functions. This was next used in the derivation of the normal forms.

Elementary singularities

The homological operators, after restriction to the spaces $\mathcal{F}_d$ of homogeneous polynomials of degree d, become endomorphisms of these spaces. We denote them by $C_d\left(\boldsymbol{V}_0\right)$ and $D_d\left(\boldsymbol{V}_0\right)$.

In the monomial basis they become diagonal:

$$\begin{aligned} C\left(\boldsymbol{V}_0\right) x^i y^j &= \left(\lambda_1 i + \lambda_2 j\right) x^i y^j, \\ D\left(\boldsymbol{V}_0\right) x^i y^j &= \left(\lambda_1 (i-1) + \lambda_2 (j-1)\right) x^i y^j. \end{aligned}$$

We use the following norm

$$\|f\| = \|f\|_\rho = \sum \left|a_{i,j}\right| \rho^{i+j}$$

of the series $f = \sum a_{i,j} x^i y^j$.

The focus case

Recall that in this case, $\lambda \notin \mathbb{R}$, the normal form is linear, because the operators $C_d\left(\boldsymbol{V}_0\right)$ and $D_d\left(\boldsymbol{V}_0\right)$ are isomorphisms. So, we can assume $\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{W}$, where $\boldsymbol{W} = O\left(|(x,y)|^D\right)$ is of high order.

To prove the analyticity of the reduction to the normal form it is enough to show that the operators $C(\boldsymbol{V})$ and $D(\boldsymbol{V})$ are invertible and that their inverses are bounded.

We have $C(\boldsymbol{V}) = C(\boldsymbol{V}_0)(I-K)$, $K = -C(\boldsymbol{V}_0)^{-1} C(\boldsymbol{W})$, and hence

$$C(\boldsymbol{V})^{-1} = \left(\sum K^n\right) C(\boldsymbol{V}_0)^{-1}.$$

We show that the series $\sum K^n$ is absolutely convergent. Similar estimates hold for $D(\boldsymbol{V})$. This is sufficient to prove the convergence of the reduction process.

The nonresonant saddle case

There exist two analytic separatrices, which can be assumed equal $\{x=0\}$ and $\{y=0\}$. Thus we can assume that the perturbation part of $\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{W}$ equals

$$\boldsymbol{W} = f_1 \boldsymbol{V}_0 + f_2 \boldsymbol{E},$$

where

$$\boldsymbol{E} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

is the standard Euler vector field. We have

$$f_1\boldsymbol{V}_0 = A(\boldsymbol{V}_0)f_1, \quad B(\boldsymbol{V}_0)\boldsymbol{W} = (\lambda_1 - \lambda_2)\,xyf_2.$$

The operators $C(\boldsymbol{V}_0)$ and $D(\boldsymbol{V}_0)$ are (formally) invertible, but we do not have good estimates for their inverses. One expects that for dense set of λ's and generic perturbation the normalizing series is divergent.

Yu. Ilyashenko proposed to consider the following 1–parameter family of perturbed vector fields

$$\boldsymbol{V}_\zeta = \boldsymbol{V}_0 + \zeta\boldsymbol{W}, \quad \zeta \in \mathbb{C}.$$

It turns out that: *either*

(1) *the normalizing series converges for all* $\zeta \in \mathbb{C}$ *in some domain* D_ρ, $\rho = \rho(\zeta) > 0$, *or*

(2) *this series diverges for all* ζ *except a set* $K_{\boldsymbol{W}}$ *of capacity zero.*

The Bogdanov–Takens singularity with nonreal λ

We want to split our homological operators into a diagonal type part and a small nilpotent type part.

To this aim we introduce new variables (x, z) such that

$$y = \varepsilon z - x^r,$$

where $\varepsilon \neq 0$ is a small constant. Then vector field $\boldsymbol{V}_H$ takes the form

$$\boldsymbol{U}_H = x^{r-1}\left(\lambda x \frac{\partial}{\partial x} + r z \frac{\partial}{\partial z}\right) + \varepsilon z \frac{\partial}{\partial x} = x^{r-1}\boldsymbol{U}_0 + \varepsilon \boldsymbol{U}_1.$$

Since the vector field $\boldsymbol{V}_H$ has invariant curves $F_1 = y + x^r = 0$ and $F_2 = y + \lambda x^r = 0$, the vector field $\boldsymbol{U}_H$ has the invariant curves

$$F_1 = z = 0 \text{ and } F_2 = \varepsilon z + (\lambda - 1)x^r = 0.$$

The normal form for Type I

We study the operators

$$C_d\left(\boldsymbol{U}_H\right), D_d\left(\boldsymbol{U}_H\right) : \mathcal{F}_d \longmapsto \mathcal{F}_{d+r-1}.$$

For $d = d_0 + rd_1$, $d_0 = 0, \cdots, r-1$, we have $\mathcal{F}_d = \text{span}\left\{x^d, x^{d-r}z, \ldots, x^{d_0}z^{d_1}\right\} \simeq \mathbb{C}^{d_1+1}$. Therefore, these operators act between:

(i) $\mathbb{C}^{d_1+1}$ and $\mathbb{C}^{d_1+2}$ if $d_0 \neq 0$, and

(ii) $\mathbb{C}^{d_1+1}$ and $\mathbb{C}^{d_1+1}$ otherwise.

Assume firstly that $\lambda > 1$ and $\lambda \notin \mathbb{N}$ (the Type I).

In Case (i) the subspace complementary to ImC_d (or ImD_d) is $1-$dimensional, previously it was chosen as $\mathbb{C} \cdot x^{d+r-1}$; in Case (ii) these operators are isomorphisms.

We have

$$C_d\left(x^{r-1}\boldsymbol{U}_0\right) x^i z^j = \left(\lambda i + rj\right) x^{r-1} \cdot x^i z^j,$$

(so, they are diagonal like), and the operators associated with $\boldsymbol{U}_1$ are rather off-diagonal.

In Case (ii) we have a easier situation, diagonal plus nilpotent. In case (i) the situation is not that clear, because the ’small’ contribution in the image in $\mathcal{F}_{d+r-1}$ is associated with $x^{\gamma}z^{\delta}$ for maximal δ.

This suggests a change in the shape of the normal form; we choose it as follows:

$$\Psi\,(x,z)\,\{\boldsymbol{U}_H + \Phi\,(x,z)\,\boldsymbol{E}_H\}\,,$$

$$\begin{aligned}\Phi &= \varphi_0(z) + x\varphi_1(z) + \ldots + x^{r-2}\varphi_{r-2}(z),\\ \Psi &= \psi_0(z) + x\psi_1(z) + \ldots + x^{r-2}\psi_{r-2}(z),\end{aligned}$$

$\boldsymbol{E}_H = x\frac{\partial}{\partial x} + rz\frac{\partial}{\partial z}$ (as before).

The analyticity

Introduce the projection operator P as follows. P, restricted to $\mathcal{F}_{d_0+rd_1}$, has the kernel $\mathbb{C}\cdot x^{d_0}z^{d_1}$ and the image spanned by the remaining monomials.

The homological operator associated with $\varepsilon\boldsymbol{U}_1$ (and its restriction to ImP) is small with respect to the one associated with $x^{r-1}\boldsymbol{U}_0$.

In order to prove the analyticity of the normal form it is enough to estimate the norms of the operators

$$C_d^{\mathrm{res}}\left(\boldsymbol{U}_H\right)^{-1}.$$

We obtain that these norm are bounded by

$$\mathrm{const}/d\rho^{r-1}.$$

The BT singularity with negative irrational λ

Here we stick to the standard coordinates x, y.

Let $F_1 = y + x^r$ and $F_2 = y + \lambda x^r$, where the curves $F_1 = 0$ and $F_2 = 0$ are invariant for $\boldsymbol{V}_H$ with the ‘cofactors’ rx^{r-1} and λrx^{r-1} respectively.

We have

$$\begin{aligned} C\left(\boldsymbol{V}_H\right) F_1^i F_2^j &= (i + \lambda j) \cdot rx^{r-1} F_1^i F_2^j, \\ D\left(\boldsymbol{V}_H\right) F_1^i F_2^j &= (i - 1 + \lambda\left(j - 1\right)) \cdot rx^{r-1} F_1^i F_2^j. \end{aligned}$$

We want to apply the arguments about nonresonant saddle singularities to our case. Therefore, we consider vector fields of the form $\boldsymbol{V}_H + \boldsymbol{W}$ and take the family

$$\boldsymbol{V}_\zeta = \boldsymbol{V}_H + \zeta \boldsymbol{W}, \quad \zeta \in \mathbb{C}.$$

The perturbation $\boldsymbol{W}$ should satisfy some conditions. The first level normal form should be trivial, i.e.,

$$\boldsymbol{W} = f\boldsymbol{V}_H + \boldsymbol{W}_1, \quad B\left(\boldsymbol{V}_H\right)\boldsymbol{W}_1 = g,$$

and $f = Pf$, $g = Qg$, where P and Q are projections onto $\mathrm{Im} C\left(\boldsymbol{V}_H\right)$ and $\mathrm{Im} D\left(\boldsymbol{V}_H\right)$ respectively.

Thus, the first level reduction relies upon applying the operators $C^{\mathrm{res}}\left(\boldsymbol{V}_H\right)^{-1}$ and $D^{\mathrm{res}}\left(\boldsymbol{V}_H\right)^{-1}$ to f and g. We write

$$f = \sum f_d, \quad g = \sum g_d,$$

where $f_d, g_d \in \mathcal{F}_d$.

We expand some summands in f and g, $(d = d_1 r)$:

$$\begin{aligned} f_{d_1 r + r - 1} &= \sum_{i+j=d_1} a_{i,j} \cdot x^{r-1} F_1^i F_2^j \\ g_{d_1 r + r - 1} &= \sum_{i+j=d_1} b_{i,j} \cdot x^{r-1} F_1^i F_2^j \end{aligned}$$

and assume that

$$a_{i,j} \neq 0, \quad b_{i,j} \neq 0 \text{ for all } i, j.$$

Our assumption about λ states that

$$\sum_{i,j} \frac{x^i y^j}{i + \lambda j} \text{ is divergent for all } (x, y) \neq (0, 0)\,.$$

The sequel proof is like in the elementary case.

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Thank you for your attention!