Meet preservers between lattices of real-valued continuous functions

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"Whenever you have to do with a structure-endowed entity Σ try to determine its group of automorphisms, the group of those element–wise transformations which leave all structural relations undisturbed. You can expect to gain a deep insight into the constitution of Σ in this way."

Hermann Weyl, 1952

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With so much structure, there are many **preserver problems** that can be considered.

Theorem (Banach[1]-Stone[2])

Let $T\colon C(X) \to C(Y)$ be a surjective, linear, and

$$\|T(f) - T(g)\| = \|f - g\|$$

for all $f, g \in C(X)$. Then $|T(1)| = 1$ and there exists a homeomorphism $\psi\colon Y \to X$ such that

$$T(f) = T(1) \cdot (f \circ \psi)$$

for all $f \in C(X)$.

[1] S. Banach, "Théorie des opérations linéarse," Chelsea, Warsaw, 1932

[2] M. Stone, "Applications of the theory of boolean rings in topology," Trans. Amer. Math. Soc., Vol. 41 (1937), pp. 375–481.

Given an $f \in C(X)$, we denote the range of f by

$$\operatorname{Ran}(f) = \{f(x) \colon x \in X\}$$

Theorem (Molnár[3])

Let $T \colon C(X) \to C(Y)$ be surjective and

$$\operatorname{Ran}(T(f) \cdot T(g)) = \operatorname{Ran}(f \cdot g)$$

for all $f, g \in C(X)$. Then $[T(1)]^2 = 1$ and there exists a homeomorphism $\psi \colon Y \to X$ such that

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[3] L. Molnár, "Some Characterizations of the Automorphisms of $B(H)$ and $C(X)$," Proc. Amer. Math. Soc., Vol. 130 (2001), pp. 111-120.

Given an $f \in C(X)$, we define

$$\mathrm{Ran}_\pi(f) = \{\lambda \in \mathrm{Ran}(f) \colon |\lambda| = \|f\|\}$$

Theorem (Hatori-Miura-Takagi[4] and Luttman-Tonev[5])

Let $T \colon C(X) \to C(Y)$ be surjective and

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for all $f, g \in C(X)$. Then $[T(1)]^2 = 1$ and there exists a homeomorphism $\psi \colon Y \to X$ such that

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[4] O. Hatori, T. Miura, and H. Takagi, "Characterization of Isometric Isomorphisms Between Uniform Algebras via Non-linear Range Preserving Properties," Proc. Amer. Math. Soc., Vol. 134 (2006), pp. 2923-2930.

[5] A. Luttman and T. Tonev, "Uniform Algebra Isomorphisms and Peripheral Multiplicativity," Proc. Amer. Math. Soc., Vol. 135 (2007), pp. 3589-3598.

Theorem (Hatori-Hirasawa-Miura[6])

Let $T\colon C(X) \to C(Y)$ *be surjective and*

$$\mathrm{Ran}_\pi(T(f) + T(g)) = \mathrm{Ran}_\pi(f + g)$$

for all $f, g \in C(X)$. *Then there exists a homeomorphism* $\psi\colon Y \to X$ *such that*

$$T(f) = f \circ \psi$$

for all $f \in C(X)$.

[6] O. Hatori, G. Hirasawa, and T. Miura, "Additively spectral-radius preserving surjections between unital semisimple commutative Banach algebras," Cent. Eur. J. Math., Vol. 8 (2010), pp. 59-601

In light of the previous results, we can posit the following:

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Question

Suppose that $\odot$ is a binary operation for continuous functions. What can be said about a surjective $T\colon C(X) \to C(Y)$ such that

$$\mathrm{Ran}_\pi(T(f) \odot T(g)) = \mathrm{Ran}_\pi(f \odot g)$$

for all $f, g \in C(X)$?

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A partial ordering can be induced on $C_{\mathbb{R}}(X)$ by using the usual ordering $\leq$ of $\mathbb{R}$ as follows:

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Note that

$$(f \vee g)(x) = \max\{f(x), g(x)\} \quad \text{and} \quad (f \wedge g)(x) = \min\{f(x), g(x)\}$$

The join $\vee$ and meet $\wedge$ are associative and commutative binary operations on $C_{\mathbb{R}}(X)$ and the set $C_{\mathbb{R}}(X)$ posses several structures related to its usual ordering:

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- $C_{\mathbb{R}}(X)$ is an *f-ring* as
  - $0 \leq f \cdot g$ whenever $0 \leq f$ and $0 \leq g$.
  - given $0 \leq f$ and g and h with $g \wedge h = 0$, then $(f \cdot g) \wedge h = 0$.

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With so much structure, there are many **preserver problems** that can be considered.

Theorem (Kaplansky[7])

Let $T\colon C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(Y)$ *be bijective, continuous, and*

$$T(f \wedge g) = T(f) \wedge T(g) \quad \text{and} \quad T(f \vee g) = T(f) \vee T(g)$$

for all $f, g \in C_{\mathbb{R}}(X)$. *Then there exists a homeomorphism* $\psi\colon Y \to X$ *and a continuous* $\varphi\colon X \times \mathbb{R} \to \mathbb{R}$ *such that*

$$T(f)(x) = \varphi(x, f(\psi(x)))$$

for all $f \in C_{\mathbb{R}}(X)$ *and* $x \in X$.

[7] I. Kaplansky, "Lattices of continuous functions", Bull. Amer. Math. Soc., Vol. 53 (1947), pp. 617-623.

Theorem (Lee)

Let $T\colon C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(Y)$ *be surjective such that*

$$\mathrm{Ran}_\pi(T(f) \wedge T(g)) = \mathrm{Ran}_\pi(f \wedge g)$$

for all $f, g \in C_{\mathbb{R}}(X)$*. Then there exists a homeomorphism* $\psi\colon Y \to X$ *such that*

$$T(f) = f \circ \psi$$

for all $f \in C_{\mathbb{R}}(X)$ *such that* $0 \le f$*.*

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- Given $f \in C_{\mathbb{R}}(X)$, we define the *maximizing set* of f by

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- Given $\alpha \in (0, \infty)$, an $h \in C_{\mathbb{R}}(X)$ is said to be an α*–peaking function* if

$$0 \leq h \quad \text{and} \quad \mathrm{Ran}_{\pi}(h) = \{\alpha\}$$

  We denote the collection of all such functions by

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- Given $\alpha \in (0, \infty)$ and $x \in X$, we define

$$\mathcal{P}_X(\alpha, x) = \{h \in \mathcal{P}_X(\alpha) \colon h(x) = \alpha\}$$

Lemma

Let $\alpha \in (0, \infty)$*, and let* $h, k \in \mathcal{P}_X(\alpha)$ *be such that* $M(h) \subset M(k)$*. Then* $M(T(h)) \subset M(T(k))$

Lemma

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Key ideas for the proof:

- For $\alpha \in (0, \infty)$, then $h \in \mathcal{P}_X(\alpha)$ if and only if $T(h) \in \mathcal{P}_Y(\alpha)$. This follows immediately from $\mathrm{Ran}_\pi(T(f) \wedge T(g)) = \mathrm{Ran}_\pi(f \wedge g)$ and the fact that $f \wedge 0 = 0$ if and only if $0 \leq f$.

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- Given $\alpha \in (0, \infty)$, an $x \in X$, and $h \in \mathcal{P}_X(\alpha)$, then $h \in \mathcal{P}_X(\alpha, x)$ if and only if $h \wedge k \in \mathcal{P}_X(\alpha)$ for all $k \in \mathcal{P}_X(\alpha, x)$.

Lemma

Let $\alpha \in (0, \infty)$*, and let* $x \in X$*. Then*

$$\bigcap_{h \in \mathcal{P}_X(\alpha, x)} M(T(h))$$

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Key idea for the proof: Combine the previous lemma with the finite intersection property and with the fact that given $\alpha \in (0,\infty)$ and $h_1,\ldots,h_n \in \mathcal{P}_X(\alpha)$ such that

$$\bigcap_{k=1}^{n} M(h_k) \neq \varnothing$$

then

$$\bigwedge_{k=1}^{n} h_k \in \mathcal{P}_X(\alpha) \quad \text{and} \quad M\left(\bigwedge_{k=1}^{n} h_k\right) = \bigcap_{k=1}^{n} M(h_k).$$

Lemma

Let $\alpha, \beta \in (0, \infty)$*, and let* $x \in X$*. Then*

$$\bigcap_{h \in \mathcal{P}_X(\alpha, x)} M(T(h)) = \bigcap_{h \in \mathcal{P}_X(\beta, x)} M(T(h))$$

Lemma

Let $\alpha, \beta \in (0, \infty)$, *and let* $x \in X$. *Then*

$$\bigcap_{h \in \mathcal{P}_X(\alpha, x)} M(T(h)) = \bigcap_{h \in \mathcal{P}_X(\beta, x)} M(T(h))$$

Key idea for the proof: Let

$$\{y\} = \bigcap_{h \in \mathcal{P}_X(\alpha, x)} M(T(h)) \quad \text{and} \quad \{z\} = \bigcap_{h \in \mathcal{P}_X(\beta, x)} M(T(h))$$

If $y \neq z$, there is an open neighborhood U of z with $y \notin U$. Choosing $k \in \mathcal{P}_Y(\beta, z)$ with $M(k) \subset U$ and scaling the $h \in \mathcal{P}_X(\beta, x)$ with $T(h) = k$ by α/β will then lead to the contradictory $y \in U$.

We now have access to the mapping $\tau\colon X \to Y$ such that

$$\bigcap_{h \in \mathcal{P}_X(\alpha, x)} M(T(h)) = \{\tau(x)\},$$

where $\alpha \in (0, \infty)$.

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Lemma

Let $x \in X$, and let $f \in C_{\mathbb{R}}(X)$ be such that $0 \leq f$. Then $T(f)(\tau(x)) = f(x)$.

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Lemma

Let $x \in X$, and let $f \in C_{\mathbb{R}}(X)$ be such that $0 \leq f$. Then $T(f)(\tau(x)) = f(x)$.

Key idea for the proof: Let $\varepsilon > 0$ and set $\delta = f(x) + \varepsilon$. We can find an $h \in \mathcal{P}_X(\delta, x)$ with $\|f \wedge h\| < \delta$. Since

$$\|T(f) \wedge T(h)\| = \|f \wedge h\|$$

and $T(h) \in \mathcal{P}_Y(\delta, \tau(x))$, it follows that

$$T(f)(\tau(x)) = \min\{T(f)(\tau(x)), T(h)(\tau(x))\} < \delta = f(x) + \varepsilon$$

Lemma

The mapping $\tau\colon X \to Y$ *is a homeomorphism.*

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The mapping $\tau : X \to Y$ is a homeomorphism.

Key idea for injectivity: If $\tau(x) = \tau(y)$ and $x \neq y$, take an open neighborhood U of x with $y \notin U$. Choosing $h \in \mathcal{P}_X(1, x)$ with $M(h) \subset U$ then yields the contradictory

$$h(y) = T(h)(\tau(y)) = T(h)(\tau(x)) = h(x) = 1$$

Lemma

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$$h(y) = T(h)(\tau(y)) = T(h)(\tau(x)) = h(x) = 1$$

Key idea for continuity: Let $V \subset Y$ be open, and let $x \in \tau^{-1}[V]$. As $\tau(x) \in V$, we can find $k \in \mathcal{P}_Y(1,\tau(x))$ with $M(k) \subset V$. There is an $h \in \mathcal{P}_X(1,x)$ with $T(h) = k$, and it can be shown that

$$\{t \in X\colon \alpha < h(t)\} \subset \tau^{-1}[V]$$

where

$$\alpha = \max\{k(y)\colon y \in Y \setminus V\} < 1$$

Key idea for surjectivity: Let $y \in Y$. Suppose that $y \notin \tau[X]$. As X is compact and τ is continuous, it follows that $U = Y \setminus \tau[X]$ is open. We can then find a $k \in \mathcal{P}_Y(1, y)$ with $M(k) \subset U$. There is an $h \in \mathcal{P}_X(1)$ with $T(h) = k$ and so any $t \in M(h)$ then satisfies

$$1 = h(t) = T(h)(\tau(t)) = k(\tau(t)),$$

which gives the contradictory

$$\tau(t) \in M(k) \subset U = Y \setminus \tau[X]$$

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$$\tau(t) \in M(k) \subset U = Y \setminus \tau[X]$$

Proof of Theorem.

Let $\psi\colon Y \to X$ be the inverse of τ. Given $f \in C_{\mathbb{R}}(X)$ with $0 \leq f$, we have $f = T(f) \circ \tau$ and so it follows that

$$T(f) = f \circ \psi$$

□

Future Work

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THANK YOU!