Regularity and finite element approximation for two-dimensional elliptic equations with line Dirac sources

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Outline

The main problem

We are interested in the regularity and the finite element method for solving the elliptic boundary value problem [H. Li, et al. 2021]

$$-\Delta u = \delta_\gamma \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

Here,

- $\Omega \subset \mathbb{R}^2$ be a polygonal domain;
- γ be a line segment strictly contained in Ω;
- δ_γ is the line Dirac measure on γ, namely,

$$\langle \delta_\gamma, v \rangle = \int_\gamma v(s)ds, \qquad \forall\, v \in L^2(\gamma).$$

Applications: monophasic flows in porous media, tissue perfusion or drug delivery by a network of blood vessels.

Literature work on FEM

γ **degenerates to a point:**

- L^2 (or H^ϵ with small ϵ) convergence [Babuška 1972, Scott 1973, 1976, Casas 1985];
- Convergence rate with graded meshes [Apel 2011];
- Optimal error estimates away from singular points in 2D and 3D [Koppl 2014].

γ **is a curve:**

- Assuming regularity in a weighted Sobolev space, optimal error estimate in 3D [DAngelo 2008, DAngelo 2012];
- Regularity later proved in [Ariche 2016];
- γ is a closed loop in 2D, element immersed interface methods [Heltai. 2019, 2020].

Challenges and Objectives

The main challenges:

- Limited regularity because of the singular source term: singular points, and singular line.
- The convergence of the finite method is slow.

The main objectives:

- Derive the regularity in a Sobolev space and weighted Sobolev space when γ is a line segment.
- Propose the finite element algorithm.
- Obtain the optimal error estimates.

Regularity in Soboelv space

Lemma

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then $\delta_\gamma \in H^{-\frac{1}{2}-\epsilon}(\Omega)$ for any $\epsilon > 0$.

Lemma

Given $\epsilon > 0$, the solution of equation (1) satisfies $u \in H^{\frac{3}{2}-\epsilon}(\Omega) \cap H_0^1(\Omega)$.

Corollary

The solution u of equation (1) is Hölder continuous $u \in C^{0,1/2-\epsilon}(\Omega)$ for any small $\epsilon > 0$. In particular, we have $u \in C^0(\Omega)$.

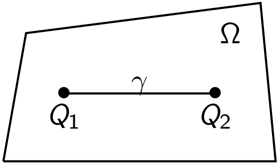


Figure: Domain Ω containing a line fracture γ.

Regularity estimates in weighted spaces

- WLOG, $\gamma = \{(x, 0),\ 0 < x < 1\}$ with the endpoints $Q_1 = (0, 0)$ and $Q_2 = (1, 0)$.
- $\mathcal{V}$: Singular set, which is the collection of Q_1, Q_2, and all the vertices of Ω.

The transmission problem Consider the equation

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega \setminus \gamma, \\ w_y^+ = w_y^- - 1 & \text{on } \gamma, \\ w^+ = w^- & \text{on } \gamma, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $w_y = \partial_y w$. Here, for a function v, $v^\pm := \lim_{\epsilon \to 0} v(x, y \pm \epsilon)$. It is clear that equation (2) has a unique weak solution

$$w \in H^1(\Omega \setminus \gamma) \cap \{w|_{\partial\Omega} = 0\}.$$

Domain decomposition

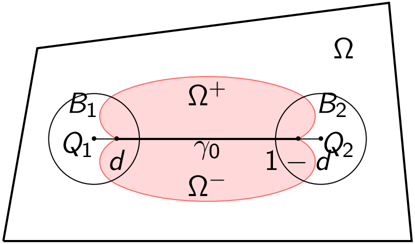


Figure: Decomposition around the singular line: Ω^+, Ω^-, B_1 and B_2.

Domain decomposition:

(i) the interior region $R_1 = \Omega^+ \cup \Omega^-$ away from the set $\mathcal{V}$;

(ii) the region $R_2 = B_1 \cup B_2$ consisting of the neighborhoods of the endpoints of γ;

(iii) $R_3 = \Omega \setminus (\bar{R}_1 \cup \bar{R}_2)$ is the region close to the boundary $\partial\Omega$ [Grisvard, 1985].

Weighted Sobolev spaces

Definition

Let $r_i(x, Q_i)$ be the distance from x to $Q_i \in \mathcal{V}$ and let

$$\rho(x) = \Pi_{Q_i \in \mathcal{V}} r_i(x, Q_i). \tag{3}$$

For $a \in \mathbb{R}$, $m \geq 0$, and $G \subset \Omega$, the weighted Sobolev space

$$\mathcal{K}_a^m(G) := \{v,\ \rho^{|\alpha|-a} \partial^\alpha v \in L^2(G), \forall\ |\alpha| \leq m\},$$

where the multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_{\geq 0}$, $|\alpha| = \alpha_1 + \alpha_2$, and $\partial^\alpha = \partial_x^{\alpha_1} \partial_y^{\alpha_2}$. The $\mathcal{K}_a^m(G)$ norm for v is defined by

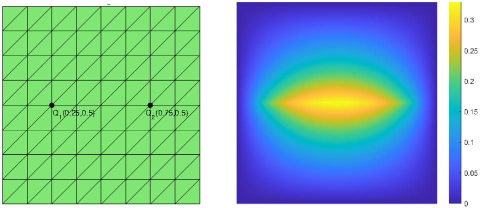
$$\|v\|_{\mathcal{K}_a^m(G)} = \Big(\sum_{|\alpha| \leq m} \iint_G |\rho^{|\alpha|-a} \partial^\alpha v|^2 dxdy \Big)^{\frac{1}{2}}.$$

In the neighborhood B_i:

$$\mathcal{K}_a^m(B_i) = \{v, r_i^{|\alpha|-a} \partial^\alpha v \in L^2(B_i), \forall\ |\alpha| \leq m\}.$$

Function space at the singular points

- Away from the set $\mathcal{V}$, the weighted space $\mathcal{K}_a^m$ is equivalent to the Sobolev space H^m;



Define

- $\chi_i \in C_0^\infty(B_i)$ that satisfies

$$\chi_i = \begin{cases} 1 & \text{in } B(Q_i, d), \\ 0 & \text{on } \partial B_i. \end{cases}$$

- the linear span of these two functions

$$W = \mathrm{span}\{\chi_i\}, \quad i = 1, 2, \tag{4}$$

Regularity in R_1 and R_2

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega \setminus \gamma, \\ w_y^+ = w_y^- - 1 & \text{on } \gamma, \\ w^+ = w^- & \text{on } \gamma, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

Lemma

The solution of equation (2) is smooth in either Ω^+ or in Ω^-. Namely, for any $m \geq 1$, $w \in H^{m+1}(\Omega^+)$ and $w \in H^{m+1}(\Omega^-)$.

Theorem

Let $B_{d,i} := B(Q_i, d) \subset B_i$, $i = 1, 2$. Then, in $B_{d,i}$, the solution w of equation (2) admits a decomposition $w = w_{reg} + w_s$, where $w_s \in W$ and $w_{reg} \in \mathcal{K}^{m+1}_{a+1}(B_{d,i} \setminus \gamma)$ for $0 < a < 1$ and $m \geq 1$. Moreover, we have

$$\|w_{reg}\|_{\mathcal{K}^{m+1}_{a+1}(B_{d,i}\setminus\gamma)} + \|w_s\|_{L^\infty(B_i)} \leq C. \qquad (5)$$

Regularity in weighted Sobolev space

Theorem

The solution u of equation (1) is smooth in the region away from the set $\mathcal{V}$, namely, for $m \geq 1$, $u \in H^{m+1}(\Omega^+)$ and $u \in H^{m+1}(\Omega^-)$. In the neighborhood of each endpoint of γ, u admits a decompostion

$$u = u_{reg} + u_s, \qquad u_s \in W,$$

such that for any $m \geq 1$ and $0 < a < 1$,

$$\|u_{reg}\|_{\mathcal{K}^{m+1}_{a+1}(B_{d,i}\setminus\gamma)} + \|u_s\|_{L^\infty(B_i)} \leq C.$$

In the region R_3 away from γ and close to the boundary, $u \in \mathcal{K}^{m+1}_{a+1}(R_3)$ for $m \geq 1$ and $0 < a < \frac{\pi}{\omega}$, where ω is the largest interior angle among all the vertices of the domain Ω.

Finite element algorithm

- $\mathcal{T} = \{T_i\}$ be a triangulation of Ω with triangles
- $S(\mathcal{T}, m) = \{v \in C^0(\Omega) \cap H_0^1(\Omega) : v|_T \in P_m(T), \ \forall \ T \in \mathcal{T}\}$, where $P_m(T)$ is polynomials with degree no more than m.
- the finite element solution $u_h \in S(\mathcal{T}, m)$ of equation (1) by

$$\int_\Omega \nabla u_h \cdot \nabla v_h dx = \int_\gamma v_h dx, \quad \forall \ v_h \in S(\mathcal{T}, m). \tag{6}$$

Error estimate on quasi-uniform meshes

- the mesh $\mathcal{T}$ consists of quasi-uniform triangles with size h
- $u \in H^{\frac{3}{2}-\epsilon}(\Omega))$, the standard error estimate [Ciarlet, 1974] yields only a sup-optimal convergence rate

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch^{\frac{1}{2}-\epsilon}, \qquad \text{for } \epsilon > 0. \tag{7}$$

Graded refinements

Algorithm (Graded refinements)

Let Q be also a vertex in a triangulation $\mathcal{T}$. Let pq be an edge in the triangulation $\mathcal{T}$ with p and q as the endpoints.

1. *(Neither p nor q coincides with Q.) We choose r as the midpoint ($|pr| = |qr|$).*
2. *(p coincides with Q.) We choose r such that $|pr| = \kappa|pq|$, where $\kappa \in (0, 0.5)$ is a parameter that will be specified later. See Figure 3 for example.*

Figure: The new node on an edge pq (left – right): $p \neq Q$ and $q \neq Q$ (midpoint); $p = Q$ ($|pr| = \kappa|pq|$, $\kappa < 0.5$).

Graded refinements (Con't)

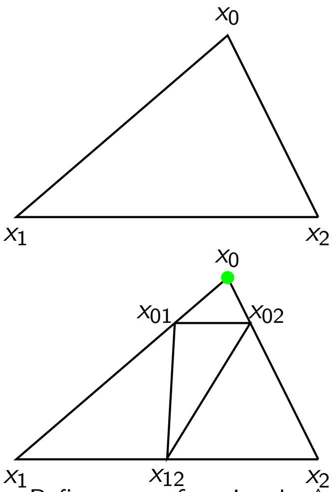


Figure: Refinement of a triangle $\triangle x_0 x_1 x_2$. First row: (left – right): the initial triangle and the midpoint refinement; second row: two consecutive graded refinements toward $x_0 = Q$, ($\kappa < 0.5$).

Optimal error estimates on graded meshes

Theorem

Recall $\kappa_Q = 2^{-\frac{m}{a}}$ *for the graded mesh on* $T_{(0)}$, $m \geq 1$ *and* $0 < a < 1$. *Let* S_n *be the finite element space associated with the graded triangulation* $\mathcal{T}_n$ *defined in Algorithm 2. Let* $u_n \in S_n$ *be the finite element solution of equation (1). Then,*

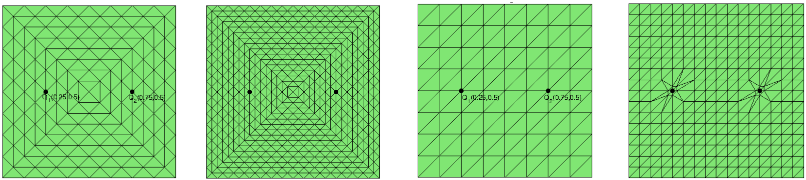
$$\|u - u_n\|_{H^1(\Omega)} \leq Ch^m \leq C\dim(S_n)^{-\frac{m}{2}},$$

where dim(S_n) *is the dimension of* S_n.

Example 1 (Union-Jack meshes and graded meshes)

Example

- square domain $\Omega = (0,1)^2$, FEM: P_1 polynomials
- $\gamma = Q_1 Q_2$ has two vertices $Q_1 = (0.25, 0.5)$ and $Q_2 = (0.75, 0.5)$



(a) (b) (c) (d)

Figure: Graded mesh and Union-Jack mesh. (a) and (b): the initial Union-Jack mesh and the mesh after one refinement. (c) and (d): the initial graded mesh and the mesh after one refinement, $\kappa = \kappa_{Q_1} = \kappa_{Q_2} = 0.2$.

Con't

Table: Convergence history with mesh refinements.

$\kappa \backslash j$	$j=2$	$j=3$	$j=4$	$j=5$
$\kappa = 0.1$	0.99	0.94	0.97	0.99
$\kappa = 0.2$	0.97	0.99	0.99	1.00
$\kappa = 0.3$	0.87	0.96	0.99	1.00
$\kappa = 0.4$	0.86	0.91	0.94	0.98
$\kappa = 0.5$	0.84	0.87	0.89	0.91
Union-Jack	0.46	0.47	0.49	0.49

- Union-Jack meshes: the convergence rate shall be about 0.5.
- Graded meshes: optimal when $\kappa := \kappa_{Q_1} = \kappa_{Q_2} = 2^{-\frac{1}{a}} < 0.5$

Example 3

- triangle domain $\Omega = \Delta ABC$ with $A = (0,0), B = (1,0)$ and $C = (0.5, 1)$, , FEM: P_2 polynomials
- $\gamma = Q_1 Q_2$ with $Q_1 = (0.3, 0.25)$, $Q_2 = (0.7, 0.25)$

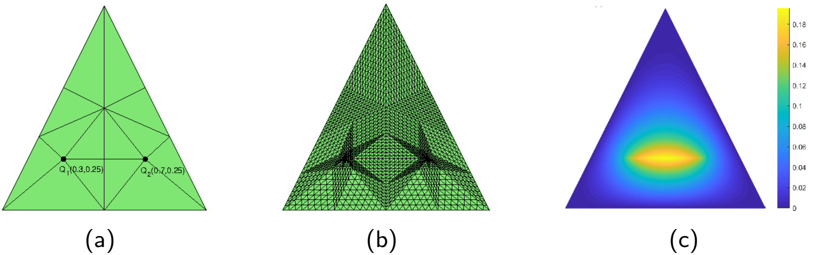


Figure: Quadratic finite element methods on graded meshes with the line fracture $\gamma = Q_1 Q_2$, $Q_1 = (0.3, 0.25)$, $Q_2 = (0.7, 0.25)$. (a) the initial mesh; (b) the mesh after four refinements, $\kappa = \kappa_{Q_1} = \kappa_{Q_2} = 0.2$; (c) the numerical solution.

Example 2

Table: Convergence history of the P_2 elements on graded meshes.

$\kappa \backslash j$	$j = 4$	$j = 5$	$j = 6$	$j = 7$
$\kappa = 0.1$	1.74	1.86	1.94	1.97
$\kappa = 0.2$	1.81	1.88	1.93	1.97
$\kappa = 0.3$	1.65	1.68	1.70	1.71
$\kappa = 0.4$	1.32	1.32	1.32	1.32
$\kappa = 0.5$	1.00	1.00	1.00	1.00

- all the interior angles of Ω are less then $\frac{\pi}{2}$, the solution is in H^3 except for the region that contains γ.
- optimal when $\kappa := \kappa_{Q_1} = \kappa_{Q_2} = 2^{-\frac{2}{a}} < 0.25$ due to the fact $0 < a < 1$

Conclusion and future plan

Conclusion

- derived the regularity in both Sobolev space and weighted Sobolev space
- Proposed a finite element algorithm.
- obtained the optimal error estimates.

Future plan

- γ is a plane in a 3D domain.
- Consider similar source term in biharmonic problem [H. Li, P. Yin, Z. Zhang].

References

Reference

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A C^0 finite element method for the biharmonic problem with Navier boundary conditions in a polygonal domain.
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