Strong Unique Continuation at the Boundary in linear elasticity and its connection with optimal stability in the determination of unknown boundaries

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Unique Continuation Property in the Interior

Let $\Omega \subset \mathbb{R}^N$ be a connected open set.
Let $\mathcal{L}$ be a linear elliptic differential operator.

$\mathcal{L}$ enjoys the **(Weak) UCP** in Ω if

$$\begin{cases} \mathcal{L}u = 0, \text{ in } \Omega, \\ u = 0 \text{ in } \Omega_0 \text{ open } \Subset \Omega \end{cases} \implies u \equiv 0 \text{ in } \Omega.$$

$\mathcal{L}$ enjoys in Ω the **SUCP** (in the Interior) if

$$\begin{cases} \mathcal{L}u = 0, \text{ in } \Omega, \\ \|u\|_{L^2(B_r(x_0))} = o(r^k), \text{ as } r \to 0, \forall k \in \mathbb{N} \end{cases} \implies u \equiv 0 \text{ in } \Omega.$$

(Here $x_0 \in \Omega$)

Strong Unique Continuation at the Boundary

$\mathcal{L}$ enjoys the **SUCPB** w.r.t. to the homogeneous boundary conditions $\mathcal{B}[u] = 0$ if

$$\begin{cases} \mathcal{L}u = 0, \text{ in } \Omega, \\ \mathcal{B}[u] = 0, \text{ on } \Sigma, \\ \|u\|_{L^2(B_r(x_0)\cap\Omega)} = o(r^k), \text{ as } r \to 0, \forall k \in \mathbb{N}, \end{cases} \qquad \Longrightarrow \qquad u \equiv 0 \text{ in } \Omega.$$

Here $x_0 \in \Sigma$, and Σ open portion (in the induced topology) of $\partial\Omega$.

UCP for $\mathcal{L} \Longrightarrow$ Propagation of Uniqueness for u

QE of UCP for $\mathcal{L} \Longrightarrow$ Propagation of Smallness for u

PROPAGATION OF SMALLNESS

$\Omega_0 \Subset \widetilde{\Omega} \Subset \Omega$

$$\begin{cases} \mathcal{L}u = 0, \text{ in } \Omega, \\ \|u\|_{L^2(\Omega_0)} \leq \varepsilon \\ \|u\|_{L^2(\Omega)} \leq 1, \end{cases} \Longrightarrow \|u\|_{L^2(\widetilde{\Omega})} \leq \omega(\varepsilon),$$

where $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Example. Three sphere inequality in the Interior: for $r_1 < r_2 < r_3$,

$$\int_{B_{r_2}(x_0)} u^2 \leq C \left(\int_{B_{r_1}(x_0)} u^2 \right)^{\theta} \left(\int_{B_{r_3}(x_0)} u^2 \right)^{1-\theta}$$

Connection between UCP and determination of unknown boundaries

A common feature in several inverse problems for PDEs concerning the determination of **an unknown boundary** Γ:

UCP for $\mathcal{L} \Rightarrow$ uniqueness for Γ

QE of UCP for $\mathcal{L} \Rightarrow$ conditional stability for Γ

QE of SUCP & SUCPB for $\mathcal{L} \Rightarrow$ optimal conditional stability for Γ

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SUCPB for **second-order elliptic equations**

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for second order parabolic equations

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- Escauriaza, Fernàndez, Vessella, Doubling properties of caloric functions, Appl. Anal. 85 (2006), no. 1–3
- Vessella, Quantitative estimates of unique continuation for parabolic equations, determination of unknown boundaries and optimal stability estimates, Inverse Problems 24 (2008), no. 2, 023001, 81 pp.

for wave equation with time independent coefficients

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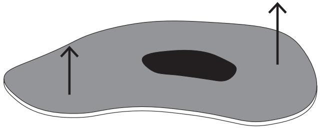
SUCPB in linear elasticity

- Kirchhoff-Love plates
  - Alessandrini, R., Vessella Optimal three spheres inequality at the boundary for the Kirchhoff-Love plate's equation with Dirichlet conditions, Arch. Rational Mech. Anal. 231 (2019), 1455–1486.
  - Morassi, R., Sincich, Vessella, Doubling inequality at the boundary for the Kirchhoff-Love plate's equation with supported conditions, Rend. Mat. Univ. Trieste, to appear.
- Generalized Plane Stress problem
  - Morassi, R., Vessella, Optimal identification of a cavity in the Generalized Plane Stress problem in linear elasticity, JEMS, to appear.

DETERMINATION OF A RIGID INCLUSION

IN A THIN ISOTROPIC ELASTIC PLATE

Thin elastic plate: $\Omega \times \left[-\frac{h}{2}, \frac{h}{2}\right]$, having middle plane Ω, D rigid inclusion



$$\mathcal{L}w := \mathrm{div}\left(\mathrm{div}\left(\mathbf{P}\nabla^2 w\right)\right) = 0, \qquad \text{in } \Omega \setminus \overline{D}.$$

where w is the **transversal displacement** and

$$\underbrace{\mathbf{P}}_{plate\ tensor} = \frac{h^3}{12}\underbrace{\mathbf{C}}_{elasticity\ tensor}$$

$$C_{ijkl} = C_{klij} = C_{klji}, \quad i,j,k,l = 1,2$$

$$\underbrace{\mathbf{C}A \cdot A \geq \gamma |A|^2,}_{Ellipticity}$$

for every 2x2 symmetric matrix A.

Assuming that the plate is made by isotropic material we have

$$\mathbf{P}A = B\left[(1-\nu)A^{sym} + \nu tr(A) I_2\right]$$

for every 2×2 matrix A, where

$$B(x) = \frac{h^3}{12}\left(\frac{E(x)}{1-\nu^2(x)}\right), \text{ (bending stiffness)}$$

$$E(x) = \frac{\mu(x)(2\mu(x)+3\lambda(x))}{\mu(x)+\lambda(x)}, \text{ (Young's modulus)}$$

$$\nu(x) = \frac{\lambda(x)}{2(\mu(x)+\lambda(x))} \text{ (Poisson's coefficient).}$$

the Lamé parameters λ, μ satisfy

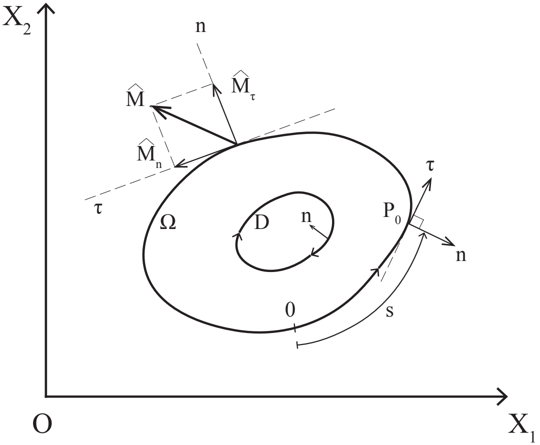
$$\mu(x) \geq \alpha_0 \quad 2\mu(x) + 3\lambda(x) \geq \gamma_0$$

Direct Problem:

$D \Subset \Omega$ rigid inclusion, D, Ω simply connected bdd domain of class $C^{1,1}$

$$(P)\begin{cases} \mathcal{L}w = 0, & \text{in } \Omega \setminus \overline{D}, \\ (\mathbf{P}\nabla^2 w)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \\ div(\mathbf{P}\nabla^2 w) \cdot n + \partial_s((\mathbf{P}\nabla^2 w)n \cdot \tau) = \partial_s(\widehat{M}_\tau), & \text{on } \partial\Omega, \\ w = 0, & \text{on } \partial D, \\ \partial_n w = 0, & \text{on } \partial D, \end{cases}$$

$\widehat{M}_\tau$ and $\widehat{M}_n$ are, respectively, the twisting and bending component of the assigned couple field $\widehat{M}$. n outer normal to $\partial(\Omega \setminus D)$, s arc length parametrization of $\partial\Omega$.



$$\widehat{M} = \widehat{M}_\tau n + \widehat{M}_n \tau = \widehat{M}_2 e_1 + \widehat{M}_1 e_2, \qquad \text{on } \partial\Omega$$

$$\tau = e_2 \times n$$

Wellposedness of the Direct Problem (P)

If Ω and D are simply connected domains of class $C^{1,1}$,
$\widehat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$, $\int_{\partial\Omega} \widehat{M}_\alpha = 0$, $\alpha = 1, 2$,
then problem (P) has a unique weak solution $w \in H^2(\Omega \setminus \overline{D})$ satisfying

$$\|w\|_{H^2(\Omega\setminus\overline{D})} \leq C\|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}.$$

INVERSE PROBLEM

Determine an **unknown** rigid inclusion D from the additional measurement of the Dirichlet data $\{w, \partial_n w\}$ taken on an open portion Σ of $\partial\Omega$, that is from the Cauchy data on Σ :

$$
(Cauchy) \begin{cases} w|_\Sigma, \\ \partial_n w|_\Sigma \\ (\mathbf{P}\nabla^2 w)n \cdot n|_\Sigma = -\widehat{M}_n \\ div(\mathbf{P}\nabla^2 w) \cdot n + \partial_s((\mathbf{P}\nabla^2 w)n \cdot \tau)|_\Sigma = \partial_s(\widehat{M}_\tau) \end{cases}
$$

APPLICATIONS

Non-destructive testing for quality assessment of materials

Hypotheses and a priori assumptions

HYPOTHESES (Concerning the Data)

- $\partial\Omega$ of class $C^{2,1}$ with constants r_0 , M_0
- $|\Omega| \leq M_1$
- $\Sigma \subset \partial\Omega$ of class $C^{3,1}$ with constants r_0 , M_0
- $\Sigma \supset \partial\Omega \cap B_{r_0}(P_0)$, for some $P_0 \in \Sigma$
- $\mathrm{supp}(\widehat{M}) \subset \Sigma$, $\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2)$, $\left(\widehat{M}_n, \partial_s(\widehat{M}_\tau)\right) \not\equiv 0$ and $\dfrac{\|\widehat{M}\|_{L^2(\partial\Omega)}}{\|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}} \leq F$
- $\lambda, \mu \in C^4(\overline{\Omega})$

A PRIORI ASSUMPTIONS (Concerning the Unknown inclusion)

- $\text{dist}(D, \partial\Omega) \geq r_0$
- ∂D of class $C^{6,\alpha}$ with constants r_0 , M_0 , $\alpha \in (0,1)$

Theorem (Stability)

Let $w_i \in H^2(\Omega \setminus \overline{D_i})$ be the solutions to (P), $i = 1, 2$.
If, given $\varepsilon > 0$, we have

$$\left\{ \|w_1 - w_2\|_{L^2(\Sigma)} + \|\partial_n (w_1 - w_2)\|_{L^2(\Sigma)} \right\} \leq \varepsilon,$$

then we have

$$d_{\mathcal{H}}(\overline{D_1}, \overline{D_2}) \leq C(|\log \varepsilon|)^{-\eta},$$

for every ε, $0 < \varepsilon < 1$, where $C > 0$, η, $0 < \eta \leq 1$, are constants only depending on the a priori data.

$d_{\mathcal{H}}(\overline{D_1}, \overline{D_2})$ is the Hausdorff distance between $\overline{D_1}$ and $\overline{D_2}$.

Morassi, R., Vessella, SIAM J. Math. Anal., 2019

Main tool of the proof

Theorem (Optimal three spheres inequality at the boundary)

If $x_0 \in \partial D$ *and*

$$\mathcal{L} w = 0, \qquad in\, \Omega \setminus \overline{D},$$

there exists $C > 1$ *such that, for every* $r_1 < r_2 < r_3 < dist(x_0, \partial\Omega)$,

$$\|w\|_{L^2(B_{r_2}(x_0)\cap(\Omega\setminus\overline{D}))} \leq C \left(\frac{r_3}{r_2}\right)^C \|w\|^{\theta}_{L^2(B_{r_1}(x_0)\cap(\Omega\setminus\overline{D}))} \|w\|^{1-\theta}_{L^2(B_{r_3}(x_0)\cap(\Omega\setminus\overline{D}))}$$

where

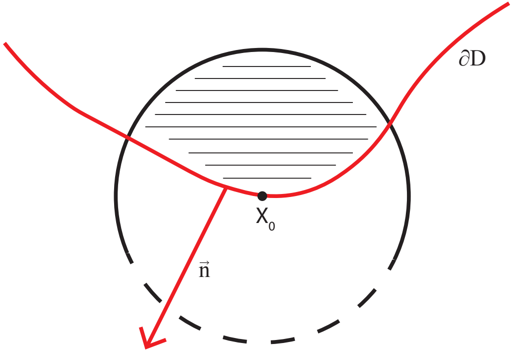
$$\theta = \frac{\log\left(\frac{r_3}{Cr_2}\right)}{\log\left(\frac{r_3}{r_1}\right)}.$$

Alessandrini, R., Vessella, ARMA, 2019

Corollary (Finite Vanishing Rate at the Boundary)

Under the above hypotheses, there exist C, N such that

$$\int_{B_r(x_0)\cap(\Omega\setminus\overline{D})} w^2 \geq Cr^N$$



In the interior, similar results hold true. In particular we have

Theorem (Finite Vanishing Rate in the Interior)

If $x_0 \in \Omega \setminus \overline{D}$ and $B_r(x_0) \Subset \Omega \setminus \overline{D}$ there exist C, N such that

$$\int_{B_r(x_0)} \left|\nabla^2 w\right|^2 \geq Cr^N$$

Basic steps of the Proof of the Optimal stability in determining rigid inclusions in K–L plates

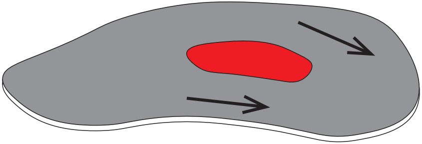
a) Stability estimates of continuation from Cauchy data:

$$\max\left\{\int_{D_1\setminus\overline{D}_2}|\nabla^2 w_2|^2,\int_{D_2\setminus\overline{D}_1}|\nabla^2 w_1|^2\right\}\leq\omega(\epsilon)$$

b) by the Finite Vanishing Rate in the Interior and at the Boundary,

$$d_{\mathcal{H}}(\overline{D_1},\overline{D_2})\leq\left(\max\left\{\int_{D_1\setminus\overline{D}_2}|\nabla^2 w_2|^2,\int_{D_2\setminus\overline{D}_1}|\nabla^2 w_1|^2\right\}\right)^{\delta}\leq(\omega(\epsilon))^{\delta}$$

GENERALIZED PLANE STRESS PROBLEM



Let D be a cavity inside the plate Ω. The in-plane displacement field $u = u_1 e_1 + u_2 e_2$ satisfies the two-dimensional Neumann problem

$$\begin{cases} \partial_\beta N_{\alpha\beta} = 0, & \text{in } \Omega \setminus \overline{D}, \\ N_{\alpha\beta} n_\beta = \widehat{N}, & \text{on } \partial\Omega, \\ N_{\alpha\beta} n_\beta = 0, & \text{on } \partial D. \end{cases} \qquad (\alpha, \beta = 1, 2)$$

where

$$N_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta}, \quad \epsilon_{\alpha\beta} = \frac{1}{2}\left(\partial_\beta u_\alpha + \partial_\alpha u_\beta\right),$$

C is the elasticity tensor of the (isotropic) material

$$\mathbf{C}A = \frac{hE(x)}{1 - \nu^2(x)}\left[(1-\nu)A^{sym} + \nu tr(A) I_2\right]$$

for every 2×2 matrix A.

Airy's function (1863)

$$\begin{cases} \partial_1 N_{11} + \partial_2 N_{12} = 0, & \text{in } \mathcal{U} := B_{R_0}(x_0) \cap (\Omega \setminus \overline{D}), \\ \partial_1 N_{21} + \partial_2 N_{22} = 0, & \text{in } \mathcal{U} := B_{R_0}(x_0) \cap (\Omega \setminus \overline{D}), \end{cases}$$

We have that

$$-N_{12}dx_1 + N_{11}dx_2, \quad -N_{22}dx_1 + N_{21}dx_2$$

are exact forms. Hence there exist $\widetilde{\varphi}_1$and $\widetilde{\varphi}_2$ such that
$(\bigstar)$ $\partial_1\widetilde{\varphi}_1 = -N_{12}$, $\partial_2\widetilde{\varphi}_1 = N_{11}$ and $\partial_1\widetilde{\varphi}_2 = -N_{22}$, $\partial_2\widetilde{\varphi}_2 = N_{21}$.
The symmetry of $N_{\alpha\beta}$ implies $N_{12} = N_{21}$, hence

$$\partial_1\widetilde{\varphi}_1 = -\partial_2\widetilde{\varphi}_2,$$

and, again, the differential form

$$-\widetilde{\varphi}_2 dx_1 + \widetilde{\varphi}_1 dx_2,$$

is exact so that there exists φ (**Airy's function**) such that

$$(\bigstar\bigstar) \qquad \partial_1\varphi = -\widetilde{\varphi}_2, \quad \partial_2\varphi = \widetilde{\varphi}_1$$

By the definition of $N_{\alpha\beta}$ and by $(\star) - (\star\star)$ we have

$$\begin{cases} \epsilon_{11} = \frac{1}{hE}\left(\partial^2_{22}\varphi - \nu\partial^2_{11}\varphi\right), \\ \epsilon_{12} = -\frac{1+\nu}{hE}\partial^2_{12}\varphi, \\ \epsilon_{22} = \frac{1}{hE}\left(\partial^2_{11}\varphi - \nu\partial^2_{22}\varphi\right) \end{cases}$$

On the other hand, since $\epsilon_{\alpha\beta} = \frac{1}{2}\left(\partial_\beta u_\alpha + \partial_\alpha u_\beta\right)$ we have

$$\partial^2_{22}\epsilon_{11} - 2\partial^2_{12}\epsilon_{12} + \partial^2_{11}\epsilon_{22} = 0$$

hence

$$\mathrm{div}\left(\mathrm{div}\left(\mathbf{L}\nabla^2\varphi\right)\right) = 0, \qquad \text{in } \mathcal{U}$$

where

$$L_{\alpha\beta\gamma\delta} = \frac{1+\nu}{hE}\delta_{\alpha\gamma}\delta_{\beta\delta} - \frac{\nu}{hE}\delta_{\alpha\beta}\delta_{\gamma\delta}$$

By a suitable choice of the primitives, we also have

$$\varphi = \partial_n \varphi = 0, \qquad \text{on } B_{R_0}(x_0) \cap \partial D$$

$$\Longrightarrow \text{ FVRB for } \varphi \text{ and for } \nabla^2 \varphi$$

$$m|\nabla^2 \varphi|^2 \leq |\widehat{\nabla u}|^2 \leq M|\nabla^2 \varphi|^2$$

Theorem (Morassi, R., Vessella, JEMS, to appear)

If ∂D is of $C^{6,\alpha}$ class and u is not constant in $B_{R_0}(x_0) \cap (\Omega \setminus \overline{D})$ then there exists C, N positive such that for every $r < R_0/2$, we have

$$\int_{B_r(x_0)\cap(\Omega\setminus\overline{D})} |\widehat{\nabla u}|^2 \geq C r^N$$

The above theorem is the main tool for the proof of optimal stability estimates for identification of cavities in the GPS problem.

SOME OPEN PROBLEMS ABOUT SUCPB IN LINEAR ELASTICITY

- **Could the assumption $\partial D \in C^{6,\alpha}$ be weakened?**
- **The case of isotropic Kirchhoff–Love plate with Neumann condition i.e. Optimal stability for unknown cavities inside the plate**
- **The case of three dimensional Lamé system**