On compact Riemann surfaces and hypermaps of genus $p+1$ where p is prime

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Compact Riemann surfaces

Definition & moduli

Definition A **Riemann surface** is a complex analytic manifold of dimension one.

Let $\mathscr{M}_g$ denote the **moduli space** of compact Riemann surfaces of genus $g \geqslant 2$.

- $\mathscr{M}_g$ has a structure of complex analytic space,
- its dimension is $3g - 3$, and
- if $g \geqslant 4$ then its singular locus is

$$\mathrm{Sing}(\mathscr{M}_g) = \{[S] : S \text{ has non-trivial automorphisms}\}$$

Equivalences

Algebraic curves & Fuchsian groups

Assume the genus to be at least two.

Theorem There is an **equivalence** between:

- compact Riemann surfaces,
- (complex projective smooth) algebraic curves,
- orbit spaces of the upper-half plane

$$\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

by the action of (co-compact) **Fuchsian groups**: discrete subgroups of

$$\mathrm{Aut}(\mathbb{H}) \cong \mathbb{P}\mathrm{SL}(2, \mathbb{R})$$

Fuchsian groups

Signature & triangle Fuchsian groups

Let Δ be a Fuchsian group such that $\mathbb{H}/\Delta$ is compact.

Definition The **signature** of Δ is the tuple

$$\sigma(\Delta) = (h; m_1, \ldots, m_l)$$

where:

- h is the **genus of the quotient** $\mathbb{H}/\Delta$ and
- $m_1, \ldots, m_l$ are the **branch indices** in the universal canonical projection

$$\mathbb{H} \to \mathbb{H}/\Delta.$$

We will be particularly interested in **triangle** Fuchsian groups: those with signature

$$(0; a, b, c) \rightarrow \text{ we simply write } (a, b, c)$$

Riemann's existence Theorem Let $S \cong \mathbb{H}/\Gamma$ be a compact Riemann surface of genus $g \geqslant 2$. A finite group G acts on S

if and only if

there is a Fuchsian group Δ and a group epimorphism (ske)

$$\theta : \Delta \to G \quad \text{such that} \quad \ker(\theta) = \Gamma$$

The group G is said to act on S with signature $\sigma(\Delta)$ and the Riemann-Hurwitz formula is satisfied

$$2(g-1) = |G|(2h - 2 + \Sigma_{j=1}^{l}(1 - 1/m_j))$$

where $\sigma(\Delta) = (h; m_1, \ldots, m_l)$.

Example

Consider a Fuchsian group of signature $(2, 2, 4, 4)$

$$\Delta = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 : \gamma_1^2 = \gamma_2^2 = \gamma_3^4 = \gamma_4^2 = \Pi_{i=1}^4 \gamma_i = 1 \rangle$$

and the **ske**

$$\theta : \Delta \to G_{5,4} = \langle a, b : a^5 = b^4 = 1, bab^{-1} = a^r \rangle$$

(r is a primitive 4-th root of 1 in $\mathbb{F}_5$) given by

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \mapsto (b^2, ab^2, ab, b^{-1}),$$

It then follows that

$$S = \mathbb{H}/\mathrm{ker}(\theta)$$

is a compact Riemann surface and

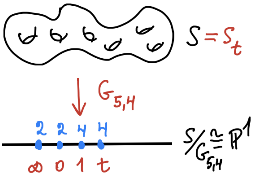
$$G_{5,4} \text{ acts on } S \text{ with signature } (2, 2, 4, 4)$$

Example

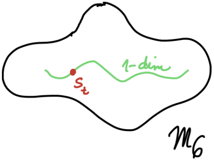
The Riemann-Hurwitz formula reads

$$2g-2 = 4\cdot 5(2\cdot 0 - 2 + 2(1-1/2) + 2(1-1/4)) \iff g = 6$$

Conclusion There exists a Riemann surface of genus $g = 6$ admitting an action of $G_{5,4}$ with signature $(2,2,4,4)$



A one-dimensional **family** of compact Riemann surfaces in the singular locus $\mathrm{Sing}(\mathscr{M}_6)$.



Admissible sequences

Definiton & general problem

Let a, b be rational numbers. The sequence

$$ag + b \quad \text{for} \quad g = 2, 3, \ldots$$

is called **admissible** if for infinitely many values of g there is a Riemann surface of genus g with a group of automorphisms of order $ag + b$.

We denote by

$$\mathscr{A}_{a,b} \subset \mathrm{Sing}(\mathscr{M}_g)$$

the set consisting of the respective surfaces.

General problem

Describe $\mathscr{A}_{a,b}$

Admissible sequences

Classical examples of admissible and well-studied sequences:

$$84g - 84, \; 8g + 8, \; 4g + 2$$

General Questions The following questions arise naturally.

- *How many* compact Riemann surfaces lie in $\mathscr{A}_{a,b}$?
- Which are the **possible groups** of automorphisms of the members of $\mathscr{A}_{a,b}$?
- Which are the **possible signatures** arising by the action of the previous groups?
- How many **different actions appear**, once the group and signature are fixed?

Notice that, in general, the set $\mathscr{A}_{a,b}$ need not be a family.

Example: the admissible sequence $4g$

Bujalance, Costa and Izquierdo[1]

- *How many* compact Riemann surfaces lie in $\mathscr{A}_{4,0}$? **complex dimension 1**
- Which are the **possible groups** of automorphisms of the members of $\mathscr{A}_{4,0}$? **the dihedral group only**
- Which are the **possible signatures** arising by the action of the previous groups? $(0;2,2,2,2g)$ **only.**
- How many **different actions appear**, once the group and signature are fixed? **only one**

Surprisingly $\mathscr{A}_{4,0}$ is a family (one stratum = the actions are all equivalent) without any additional condition on g.

1 E. Bujalance, A. F. Costa and M. Izquierdo, *On Riemann surfaces of genus g with $4g$ automorphisms*, Topology Appl. **218** (2017) 1–18.

The case $\mathscr{A}_{a,-a}$

The possibilities for Riemann surfaces S of genus g and their automorphism groups depend heavily on the factorisation of

$$\chi(S) = 2 - 2g$$

The **simplest case** of admissible sequence to consider is

$ag - a$ on the assumption that $q := g - 1$ is prime.

This problem was first considered by Belolipetsky and Jones.[2]

They restrict to the case $a \geqslant 7$ and q sufficiently large (to avoid sporadic cases) and proved that $\mathscr{A}_{a,-a}$ "splits" into three infinite series of **quasiplatonic** surfaces.

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[2] M. V. Belolipetsky and G. A. Jones, *Automorphism groups of Riemann surfaces of genus $p+1$, where p is prime.* Glasg. Math. J. **47** (2005), no. 2, 379–393.

The case $\mathscr{A}_{a,-a}$

The subcase $a = 4$

A group of automorphisms G of a surface of genus g is **large** if

$$|G| > 4g - 4$$

In this case, the surface is either quasiplatonic or belongs to a one-dimensional family such that the signature of the action is

$$(0;2,2,2,n) \text{ for } n \geqslant 3 \text{ or } (0;2,2,3,n) \text{ for } 3 \leqslant n \leqslant 5.$$

The case $4g - 4$ is therefore the **"maximal non-large"** case. These surfaces were recently considered[3]. Indeed, $\mathscr{A}_{4,-4}$ consists of:

- a two-dimensional family for each g, and
- a one-dimensional family (if $g \equiv 2 \bmod 4$).

[3] S. R-C., *On Riemann surfaces of genus g with $4g-4$ automorphisms*, Israel J. Math. **237**, 415–436 (2020).

The case $\mathscr{A}_{a,-a}$

The subcases $a = 3, 5, 6$

The previous results were recently extended[4] to the cases

$$3(g-1),\ 5(g-1)\ \text{ and }\ 6(g-1).$$

- As in the case $a = 4$, for $a = 3, 6$ appear positive dimensional families of surfaces.
- The surfaces found for $a = 5$ agree with the ones with $a = 10$ obtained earlier.
- The ones obtained for $a \geqslant 7$ appear as special points in these families.

[4] M. Izquierdo and S. R-C., *A note of large automorphism groups of compact Riemann surfaces*, J. Algebra **547** (2020), 1–21.

$\mathscr{A}_{a,-a}$

The classification is now complete

Theorem 1

Let S be a compact Riemann surface of genus

$$g = q + 1 \text{ for some prime } q \geqslant 7.$$

There is a subgroup $G \leqslant \mathrm{Aut}(S)$ of order

$$|G| = a(g-1) = aq \text{ for some integer } a \geqslant 3$$

if and only if one of the following holds.

Notation: let r be a primitive n-th root of unity in $\mathbb{F}_q$. Write:

$$G_{q,n} := \langle a, b : a^q = b^n = 1, bab^{-1} = a^r \rangle = C_q \rtimes_n C_n$$

case	a	signature	group	$q \equiv$	surfaces
(i)	12	$(2,6,6)$	$G_{q,6} \times C_2$	$1(3)$	$S_1, \bar{S}_1$ *
(ii)	10	$(2,5,10)$	$G_{q,10}$	$1(5)$	$S_2, \bar{S}_2, S_2', \bar{S}_2'$
(iii)	8	$(2,8,8)$	$G_{q,8}$	$1(8)$	$S_3, \bar{S}_3$
(iv)	6	$(3,6,6)$	$G_{q,6}, G_{q,3} \times C_2$	$1(3)$	$S_1, \bar{S}_1$
(v)	6	$(2,2,3,3)$	$G_{q,6}$	$1(3)$	$\mathscr{C}_1$
(vi)	5	$(5,5,5)$	$G_{q,5}$	$1(5)$	$S_2, \bar{S}_2, S_2', \bar{S}_2'$
(vii)	4	$(2,2,4,4)$	$G_{q,4}$	$1(4)$	$\mathscr{C}_1'$
(viii)	4	(2^5)	$\mathbf{D}_{2q}$	none	$\mathscr{C}_2$
(ix)	3	$(3,3,3,3)$	$G_{q,3}$	$1(3)$	$\mathscr{C}_1$
(x)	84	$(2,3,7)$	$\mathbb{P}\mathrm{SL}(2,13)$	13	Y_1, Y_2, Y_3
(xi)	48	$(2,3,8)$	$\mathbb{P}\mathrm{GL}(2,7)$	7	X_1, X_2
(xii)	24	$(3,3,4)$	$\mathbb{P}\mathrm{SL}(2,7)$	7	X_1, X_2

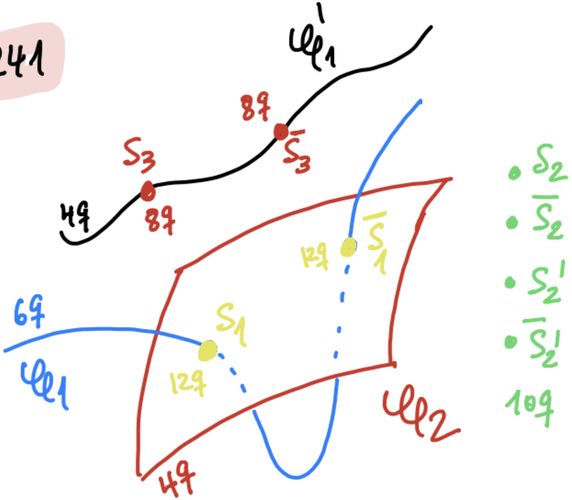
Remarks

- The actions are explicitly given in terms of surface-kernel epimorphisms.
- The table also gives the **full** automorphism groups.
- The cases $q = 2, 3, 5$ and $a = 1, 2$ lead to less uniform behavior (but they are well-known).
- **Corollary.** There is no a compact Riemann surface of genus $g = q + 1$ (q prime) with exactly

  $$3(g-1) \text{ or } 5(g-1)$$

  automorphisms!
- The families $\mathscr{C}_1, \mathscr{C}_1'$ and $\mathscr{C}_2$ are **equisymmetric**, namely, there is only one class of topological action.

$q = 241$



in $\mathrm{Sing}(\mathcal{M}_{242})$

Belyi pairs

Definition A Riemann surface S is called a **Belyi surface** if

$$\exists \beta : S \to \mathbb{P}^1 \text{ holomorphic with three critical values.}$$

The pair (S, β) is called a **Belyi pair**.

Amongst Belyi pairs, the **regular** ones are those for which

$$\beta : S \to S/G \cong \mathbb{P}^1 \text{ is given by the action of } G \leqslant \mathrm{Aut}(S).$$

Equivalently:

- S is quasiplatonic (rigid in the moduli space)
- S is uniformised by a finite index normal subgroup of a triangle Fuchsian group $\Delta(a, b, c)$

Dessin d'enfants

Definition A **dessin d'enfant** is an embedding of a connected, **bipartite** graph

$$\mathscr{G} \hookrightarrow X$$

on an **oriented** compact topological surface X such that the components of $X - \mathscr{G}$ are homeomorphic to open discs.

The dessin d'enfant is called **regular** if its automorphism group acts transitively on their edges.

Roughly speaking, there is a **correspondence** between

- (regular) dessin d'enfants,
- (regular) Belyi pairs, and
- algebraic curves defined over number fields.

Grothendieck, Belyi, Shabat, Singerman, Jones, Wolfart,...

Dessin d'enfants = hypermaps

A special kind of dessin d'enfants are those that have "black" vertices all with degree 2:

- **clean** dessin d'enfants,
- **maps** (forget the colours!)
- **platonic surfaces**, namely, uniformised by finite index subgroups of Fuchsian groups of signature $\Delta(2, b, c)$

Thus, for the **regular** case:

regular hypermap ≡ regular dessin d'enfant ≡

≡ regular Belyi pair ≡ quasiplatonic surface

Theorem 2

The orientably regular maps/hypermaps (or, equivalently, regular dessin d'enfants on Riemann surfaces) of genus

$$g = q + 1 \text{ for some prime } q \geqslant 7$$

with orientation-preserving automorphism group G of order divisible by q, are given in the following table.

- up to duality/triality, permuting the roles of vertices, edges and faces.
- N is the number of orientably regular maps/hypermaps supported by the surfaces.

$\mathscr{A}_{a,-a}$

The classification of dessins d'enfant or maps/hypermaps

case	a	type	group	$q \equiv$	surfaces	N
(i)	12	$\{6,6\}$	$G_{q,6} \times C_2$	$1(3)$	$S_1, \bar{S}_1$	1cp
(ii)	10	$\{5,10\}$	$G_{q,10}$	$1(5)$	$S_2, \bar{S}_2, S_2', \bar{S}_2'$	2cp
(iii)	8	$\{8,8\}$	$G_{q,8}$	$1(8)$	$S_3, \bar{S}_3$	1cp
(iv)	6	$(3,6,6)$	$G_{q,6}$	$1(3)$	$S_1, \bar{S}_1$	1cp
(iv)'	6	$(3,6,6)$	$G_{q,3} \times C_2$	$1(3)$	$S_1, \bar{S}_1$	1cp
(vi)	5	$(5,5,5)$	$G_{q,5}$	$1(5)$	$S_2, \bar{S}_2, S_2', \bar{S}_2'$	12t
(x)	84	$\{3,7\}$	$\mathbb{P}\mathrm{SL}(2,13)$	13	Y_1, Y_2, Y_3	3t
(xi)	48	$\{3,8\}$	$\mathbb{P}\mathrm{GL}(2,7)$	7	X_1, X_2	2t
(xii)	24	$(3,3,4)$	$\mathbb{P}\mathrm{SL}(2,7)$	7	X_1, X_2	2t

Final comments

- The small cases are identified in Conder's list.
- A similar classification for the non-orientable case of characteristic $-q$ is derived (the associated orientable double cover are the cases (x), (xi) and (xii)). This is related to earlier work of Conder, Širáň and Tucker[5].
- We also provide isogeny decomposition of the associated Jacobian varieties with group action. For instance, the surfaces in case (i) decompose as

$$JS \sim E \times E' \times JX^6$$

where E, E' are elliptic curves and X is a quotient of S.

[5] M. Conder, J. Širáň and T. Tucker, *The genera, reflexibility and simplicity of regular maps*. J. Eur. Math. Soc. (JEMS) **12** (2010), no. 2, 343–364.

Further details

M. Izquierdo, G. A. Jones and S. R-C. *Groups of automorphisms of Riemann surfaces and maps of genus $p+1$ where p is prime*, To appear in Ann. Acad. Sci. Fenn. Math. arXiv:2003.05017

Thanks! - Hvala!