On the Connectivity of Branch Loci of Spaces of Curves

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Joint work with A. Costa and other (important) people

Given an orientable, **closed** surface X of genus $g \geq 2$ The equivalence:

$(X, \mathcal{M}(X)$, complex atlas) ($\mathcal{M}(X) = \langle x, y \rangle$, $p(x, y) = 0$, the field of meromorphic functions on X)

$X \cong \frac{\mathbb{H}}{\Delta}$, with Δ a (cocompact) **Fuchsian group**
Δ discrete subgroup of $PSL(2, \mathbb{R})$

$(X, \mathcal{M}(X)$, **complex curve**) ($\mathcal{M}(X) = \mathbb{C}[x, y]/p(x, y)$, the field of rational functions on X)

(Y, dianalytic atlas) $\cong$ ($X/\overline{\sigma}$, $\overline{\sigma}$ class of anticonformal involution) $\cong$ real curve (Y, birational structure). $Y \cong \frac{\mathbb{H}}{\widehat{\Delta}}$, with $\widehat{\Delta}$ an **NEC group**

The ovals of the curve Y are the boundary components of the surface $X/\overline{\sigma}$, the orientability is the one of $X/\overline{\sigma}$, the genus (is the genus): **topological type t**

(X, complex atlas) $\cong \mathbb{H}/\Delta$, with Δ a (cocompact) Fuchsian group

Surface Fuchsian Group $\Gamma_g = \langle a_1, b_1, \ldots, a_g, b_g \mid \Pi[a_i, b_i] = 1\rangle$

- **Teichmüller** space $\mathcal{T}_g$, space of geometries on a surface of genus g
  $\mathcal{T}_g = \{\sigma : \Gamma_0 \to PSL(2, \mathbb{R}) \mid \sigma \, injective, \, \sigma(\Gamma_0) \text{ discrete } \}/PSL(2, \mathbb{R})$
  A Riemann surface with prescribed geometry is given by a marked polygon (and all its conjugate by a hyperbolic transformation) in the hyperbolic plane, or the space of conjugacy classes of Fuchsian groups isomorphic to the abstract group $\Gamma_0 = \langle a_1, b_1, \ldots, a_g, b_g;\ a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1\rangle$.
- **Moduli** space $\mathcal{M}_g$, space (orbifold) of conformal structures on a surface of genus g
- **Mapping Class Group** (Teichmüller Modular Group)
  $M_g^+ = Diff^+(X)/Diff_0(X) = Out(\Gamma_g)$
- **Orbifold Universal Covering** $\mathcal{M}_g = \mathcal{T}_g / M_g^+$
  $\mathcal{B}_g$ **Branch Locus = Singular Locus of $\mathcal{M}_g$ as orbifold** (**Not** the singular set of $\mathcal{M}_{2,3}$ as algebraic variety, A. Costa- A. Porto for a proof with Fuchsian groups)

Nielsen Realization Theorem (Abikoff 1980, Macbeath for NEC groups)

$\mathcal{B}_g = \{X \in \mathcal{M}_g \mid Aut(X) \neq 1\}$

($\mathcal{B}_2$: surfaces with more automorphisms than the hyperelliptic involution)

$g = 1$ Euclidean case: $\mathcal{T}_1 = \mathbb{H}, M_1 = PSL(2,\mathbb{Z}), \mathcal{B}_1 = \{i, e^{i\pi/3}\}$, $\mathcal{M}_1$ hyperbolic triangle with a vertex at ∞, the nodal curve $y^2 = x^3$.

Considering (X, dianalytic atlas, top. type $\mathbf{t}$) $\cong \mathbb{H}/\widehat{\Delta}$, with $\widehat{\Delta}$ an NEC group
$\mathcal{T}_t^K$ and $\mathcal{M}_t^K$ the Teichmüller and moduli space of Klein surfaces of topological type $\mathbf{t}$.

$\mathcal{M}_t^K = \mathcal{T}_t^K / M(\widehat{\Delta})$, $\quad M(\widehat{\Delta}) = Out(\widehat{\Delta})$. Branch locus $\mathcal{B}_t^K$

Studies of branch locus and moduli spaces:

For $g = 1$ Schwarz

For $g = 2$ Bolza (1887, moduli of automorphic functions)
For hyperbolic surfaces Harvey, Natanzon, Macbeath.

Deligne-Mumford Complection (going to ∞ in $\mathcal{M}_g$)

Curves whose singularities are ordinary double points (nodes), all of whose irreducible components isomorphic to $\mathbb{P}^1$ (or $\widehat{\mathbb{C}}$), meet the other irreducible components in at least 3 nodes: stable curves

$\widehat{\mathcal{M}}_g = \mathcal{M}_g \cup \{\text{stable curves}\}$

(deforming by variating the coeffcients or roots)

Geometrically: Riemann surfaces with a geodesic multicurve pinched to length 0

(deforming by variating the lengths of a system of curves)

Consider the complection $\widehat{\mathcal{B}}_g$ of $\mathcal{B}_g$ in $\widehat{\mathcal{M}}_g$

Wish: If $\mathcal{B}_g, \mathcal{B}_t^K, \widehat{\mathcal{B}}_g$ connected one can deform a curve with symmetry to another curve with symmetry along a path of curves, all they with symmetry, maybe pinching some multicurve.

1. The branch loci $\mathcal{B}_g$ of moduli spaces of hyperbolic Riemann surfaces are disconnected for all genera with the exception of genera **3**, **4**, **7**, **13**, **17**, **19** and **59**.
   Bartolini-Costa-I 2013 (Ann. Acad. Sci. Fenn.)
   In genus 2 Wiman's curve (of type I) is isolated.
2. It constains several connected components. E.g. $\mathcal{B}_g$ contains isolated strata formed by p-gonal RS for genera a multiple g of (p-1)/2, at least 2(p-1)/2
   Bartolini-Costa-I-Porto 2010/2012 (RACSAM), Costa-I 2011 (Math. Scand.)
   Question: How much does the no. of connected comp. grow?
3. Considering RS as Klein surfaces, $\mathcal{B}^K_{(g,+,0)}$ is connected!
   Bartolini-Costa-I-Porto 2010 (RACSAM)
4. $\mathcal{B}^K_{(g,+,k)}$ is connected (orientable Klein surfaces) Costa-I-Porto 2015 (Geom. Dedic.)
5. $\mathcal{B}^K_{(g,-,0)}$ is connected ($g = 4, 5$ Bujalance-Etayo-Martínez-Szpietowski 2014)
   In general? (Costa-I-Porto 2021).

6 Considering $\widehat{\mathcal{M}}_g$,

- Question1: Is $\widehat{\mathcal{B}}_g$ connected?
- Question 2: Is the locus of stable p-gonal curves connected, p odd prime?

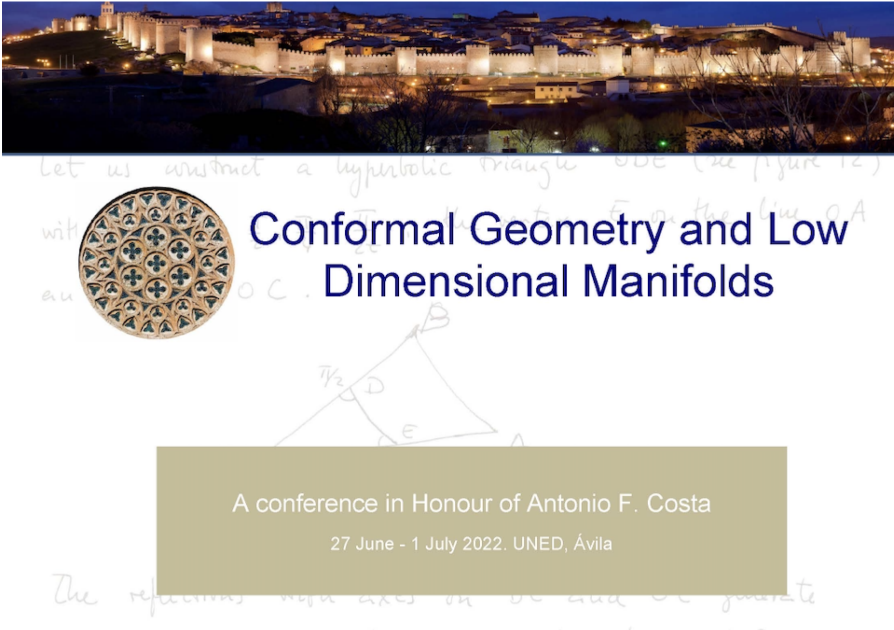
7 The hyperelliptic locus is connected (Seppälä 1982), the p-gonal locus is in general disconnected, each connected comp. associated to a partition of 0modp (González-Diez 1995, Buser-Silhol-Seppälä 1995)

8 The locus of hyperelliptic non-orientable Klein surface with one boundary component is disconnected. It is connected for the corresponding orientable surfaces.
Costa-I-Porto 2017 (Inter. J. Math.)

9 The complection of the trigonal locus is connected
Costa-I-Parlier 2014 (Rev. Mat. Complut.)

10 $\widehat{\mathcal{B}}_g$ contains isolated strata of dim.1 for genera $g = p - 1, p \geq 11$. These strata consists of p-gonal curves
Costa-I-Parlier 2014 (Rev. Mat. Complut.)

11 The locus of principally polarized abelian varieties (ppav) admitting involutions is connected
Reyes-Carocca - Rodríguez 2018


Conformal Geometry and Low Dimensional Manifolds
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Fuchsian and NEC Groups

- Δ (cocompact) discrete subgroup of $PSL(2,\mathbb{R})$
- A (compact) Riemann (surface) orbifold of genus $g \geq 2$ $\quad X = \frac{\mathbb{H}}{\Delta}$
- Δ has presentation:
  generators: $x_1, ..., x_r, a_1, b_1, ..., a_h, b_h$
  relations: $x_i^{m_i}, i = 1 : r,\ x_1...x_r a_1 b_1 a_1^{-1} b_1^{-1} ... a_h b_h a_h^{-1} b_h^{-1}$
- $X = \frac{\mathbb{H}}{\Delta}$: orbifold with r cone points and underlying surface of genus g
- Algebraic structure of Δ and geometric structure of X are determined by the signature $\quad s(\Delta) = (h; m_1, \ldots, m_r)$
- NEC group Δ (hyperbolic silvered 2-orbifolds)
- extra generators: $e_1, \ldots, e_k,\ c_{i,j},\ 1 \leq i \leq k, 1 \leq j \leq r_i + 1$
  extra relations: $(c_{i,j-1} c_{i,j})^{n_{i,j}},\ j = 1, ..., r_i,\ e_i^{-1} c_{i,r_i} e_i^{-1} c_{i,0}, i = 1, ..., k,$
  long relation: either $x_1...x_r e_1...e_k a_1 b_1 a_1^{-1} b_1^{-1} ... a_h b_h a_h^{-1} b_h^{-1}$
  or $x_1...x_r e_1...e_k d_1^2 ... d_h^2,$
- $s(\Delta) = (h; \pm; [m_1, ..., m_r]; \{(n_{1,1}, ..., n_{1,r_1}), ..., (n_{k,1}, ..., n_{k,r_k})\}).$

Singerman 1970-1974

Fundamental polygon

- Area of Δ: area of a fundamental region P

  $\mu(\Delta) = 2\pi(2h - 2 + \sum_1^r (1 - \frac{1}{m_i}))$

- For NEC group
  $\mu(\Delta) = 2\pi(\varepsilon h - 2 + k + \sum_{i=1}^r (1 - \frac{1}{m_i}) + \frac{1}{2}\sum_{i=1}^k \sum_{j=1}^{r_i} (1 - \frac{1}{n_{i,j}}))$,

- X hyperbolic equivalent to $P/\langle \text{pairing} \rangle$

- Every Riemann/Klein orbifold is diconformally equiv. to a Riemann/Klein surface X (uniformized by a surface group $\Gamma_g, \Gamma_{(g,\pm,k)}$) Moore 197X, Bujalance 1982, (Armstrong 1984 for structures associated to more general discontinuos groups)

Automorphisms and Morphisms of RS

G finite group of automorphisms of $X_g = \mathbb{H}/\Gamma$, Γ a surface group iif there exist Δ Fuchsian/NEC group and epimorphism $\theta : \Delta \to G$ with $Ker(\theta) = \Gamma$

θ is the monodromy of the (regular) covering $f : \mathbb{H}/\Gamma \to \mathbb{H}/\Delta$

$$\begin{array}{ccc} & & \mathbb{H} \\ & \swarrow & \\ X = \mathbb{H}/\Gamma & & \downarrow \\ & \searrow & \\ & & X/G = \mathbb{H}/\Delta \end{array}$$

Δ: lifting to $\mathbb{H}$ of G

An automorphism of X will fix the class of the uniformizing Fuchsian/NEC group

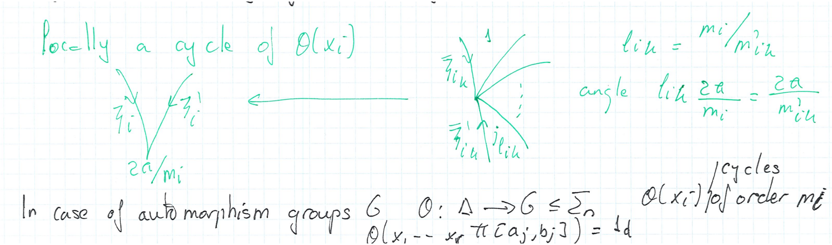
A morphism $f : X = \mathbb{H}/\Lambda \to Y = \mathbb{H}/\Delta$, given by the group inclusion $i : \Lambda \to \Delta$
Covering f determined by monodromy $\theta : \Delta \to \Sigma_{|\Delta:\Lambda}, \Lambda| = \theta^{-1}(STb(1))$
(symbol $\leftrightarrow$ Λ-coset $\leftrightarrow$ sheet for f)

Theorem (Singerman 1971) Λ (and so i) determined θ (and Δ): If $s(\Delta) = (h; m_1, \ldots, m_r)$, then $s(\Lambda) = (h'; m'_{11}, \ldots, m'_{1s_1}, \ldots, m'_{r1}, \ldots, m'_{rs_r})$ iff $\theta : \Delta \to \Sigma_{|\Delta:\Lambda|}$ s.t.

i) Riemann-Hurwitz $\frac{\mu(\Lambda)}{\mu(\Delta)} = |\Delta : \Lambda|$

ii) $\theta(x_i)$ product of s_i cycles each of length $\frac{m_i}{m'_{i1}}, \ldots, \frac{m_i}{m'_{is_i}}$

Analogous result for NEC group & Klein surfaces Singerman 1974, Hoare 1990, Pride 1990

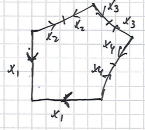


Example: Surfaces of genus 2 with 8 automorphisms. They admit an action of D_4 with monodromy $\theta: \Delta(0; 2,2,2,4) \longrightarrow D_4$

$\theta(x_1) = a = (1,3,5,7)(2,4,6,8)$

$\theta(x_2) = s = (1,2)(4,7)(3,8)(6,5)$

$\theta(x_3) = sa = (1,4)(2,3)(5,8)(6,7)$

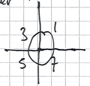


$X/_2 = \mathbb{H}/\Delta$

(of course $\theta(x_4) = a^2 = (1,5)(2,6)(3,7)(4,8)$)

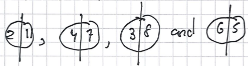
No singular pts

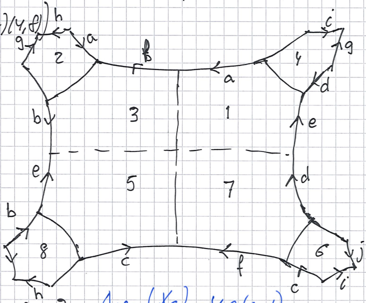
for order 4



and

for one of order 2





The area is $2\pi 8(\frac{1}{4}) = 4\pi$, so genus is 2 $\quad$ $\text{Area}(X_g) = 4\pi(g-1)$

p-gonal Riemann Surfaces

- A Riemann surface X is called *p-gonal* if it admits a morphism of degree p on the Riemann sphere
- X is called *cyclic p-gonal* when X has an automorphism φ of order p such that $X/\langle\varphi\rangle = \hat{\mathbb{C}}$.
- Case $p = 2$: X hyperelliptic R.S.
- A Riemann surface X is called *elliptic-p-gonal* if it admits a morphism of degree p on a torus.
- X is called *cyclic elliptic-p-gonal* when the morphism is a regular covering.
- *Severi-Castelnuovo inequality*: A p-gonal morphism of X is unique if the genus of $X \geq (p-1)^2$.
- An elliptic-p-gonal morphism of X is unique if the genus of $X \geq 2p + (p-1)^2$.

Teichmüller and Moduli Spaces

Δ abstract Fuchsian group $\quad s(\Delta) = (h; m_1, \dots, m_r)$

$\mathcal{T}_\Delta = \{\sigma : \Delta \to PSL(2,\mathbb{R}) \mid \sigma\, injective,\ \sigma(\Delta) \text{ discrete } \}/PSL(2,\mathbb{R})$

Teichmüller space $\mathcal{T}_\Delta$ has a complex structure of dim $3h - 3 + r$, diffeomorphic to a ball of dim $6h - 6 + 2r$.

If Λ subgroup of Δ $(i : \Lambda \to \Delta) \Rightarrow i_* : \mathcal{T}_\Delta \to \mathcal{T}_\Lambda$ embedding

Γ_g surface Fuchsian group $\quad \Gamma_g \leq \Delta \quad \mathcal{T}_\Delta \subset \mathcal{T}_{\Gamma_g} = \mathcal{T}_g$

G finite group $\quad \mathcal{T}_g^G = \{[\sigma] \in \mathcal{T}_g \mid g[\sigma] = [\sigma] \forall g \in G\} \neq \emptyset$

$\mathcal{T}_g^G$: surfaces with G as a group of automorphisms.

Mapping class group $M^+(\Delta) = Out(\Delta) = \frac{Diff(\mathbb{H}/\Delta)}{Diff_0(\mathbb{H}/\Delta)}$

$\Delta = \pi_1(\mathbb{H}/\Delta)$ as orbifold

$M^+(\Delta)$ acts properly discontinuously on $\mathcal{T}_\Delta \quad \mathcal{M}_\Delta = \mathcal{T}_\Delta / M^+(\Delta)$

- We can give coordinates to this space by considering decomposition in **pairs of pants: Fenchel-Nielsen Coordinates**.
- A **pairs of pants** is a surface with boundary obtained by taking two identical copies of a right-angle hexagon and gluing 3 of the sides. A pair of pants is homeomorphic to a sphere with three holes, the boundaries are totally geodesic (any point on the boundary has a neighbourhood isometric to a half-disc). Given three positive real numbers l_1, l_2, l_3, there is a pair of pants whose boundaries have lengths l_1, l_2, l_3 respectively.
- Any hyperbolic surface S_g admits a decomposition in $2g-2$ pairs of pants with $3g-3$ boundaries (there are many such decompositions)
- So we have $3g-3$ parameters that are the lengths of the boundaries in the pant decompositions $(l_1, l_2, \dots l_{3g-3}, \dots\dots)$. The remaining $3g-3$ parameters $\theta_1, \dots \theta_{3g-3}$ are the **twist parameters**, each one giving the angle along which two pairs of pants are glued together along the common boundary.

$$(l_1, l_2, \dots l_{3g-3}, \theta_1, \dots \theta_{3g-3})$$

- (Teichmüller) In fact the map asinging to each class of triples the Fenchel-Nielsen parameters is a homeomorphism $\mathcal{T}_g \to \mathbb{R}^{6g-6}$.
- This map is not only a homeomorphism but also a conformal map $\mathcal{T}_g \to \mathbb{C}^{3g-3}$. (Beltrami, Ahlfors).

Surfaces with automorphisms: **Branch Locus**

Consider a marked surface $\sigma(X) \in \mathcal{T}_g$ and $\beta \in M_g^+$, we have

$$\begin{array}{ccc} \mathbb{H}/\Delta_g = X & \stackrel{\sigma}{\to} & \sigma(X) \\ \downarrow & & \downarrow \\ \beta_*(X) & \stackrel{\sigma}{\to} & \sigma\beta(X) \end{array} \quad \text{biconformal}$$

$\beta[\sigma] = [\sigma] \qquad \Leftrightarrow \qquad \gamma \in PSL(2,\mathbb{R}), \quad \sigma(\Gamma_g) = \gamma^{-1}\sigma\beta(\Gamma_g)\gamma$

γ induces an automorphism of $[\sigma(X)]$

$Stb_{\mathcal{M}_g}[\sigma] = \{\beta \in M_g \mid \beta[\sigma] = [\sigma]\} = Aut([\sigma(X)])$

$G = Aut(X)$ finite, determines a conjugacy class of finite subgroups of M_g, the **symmetry** of X

X_g, Y_g equisymmetric if $Aut(X_g)$ conjugate to $Aut(Y_g)$

($Aut(X_g)$: **full automorphism group**)
Singerman's list of non-maximal signatures.

Equisymmetric Stratification

Action: $\theta : \Delta \to Aut(X_g) = G,\ ker(\theta) = \Gamma_g$

$Aut(X_g) = G$ conjugate $Aut(Y_g)$ iff $w \in Aut(G), h \in Diff^+(X)$
$\epsilon, \epsilon' : G \to Diff^+(X), \epsilon'(g) = h\epsilon w(g)h^{-1}$

Two (surface) monodromies $\theta_1, \theta_2 : \Delta \to G$ topologically equiv. actions of G

$$\beta \in Aut(\Delta) \quad \begin{array}{ccc} \Delta & \xrightarrow{\theta_1} & G \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{\theta_2} & G \end{array} \quad w \in Aut(G)$$

$\theta_1,\ \theta_2$ equiv under $\mathcal{B}(\Delta) \times Aut(G)$, $\mathcal{B}(\Delta)$ **braid group**

Broughton (1990): **Equisymmetric Stratification**

$\mathcal{M}_g^{G,\theta} = \{X \in \mathcal{M}_g \,|\, \text{symmetry type of } X \text{ is } G\}$

$\overline{\mathcal{M}}_g^{G,\theta} = \{X \in \mathcal{M}_g \,|\, \text{symmetry type of } X \text{ contains } G\}$

$\mathcal{M}_g^{G,\theta}$ smooth, connected, locally closed alg. var. of $\mathcal{M}_g$, dense in $\overline{\mathcal{M}}_g^{G,\theta}$

$$\mathcal{B}_g = \cup \overline{\mathcal{M}}_g^{G,\theta}$$

Costa-I (2008) $\mathcal{B}_g = \cup \overline{\mathcal{M}}_g^{C_p,\theta}$ (Cornalba 1987 and 2008)

Connectedness, we are interested in $Y \in \overline{\mathcal{M}}_g^{G_1,\theta_1} \cap \overline{\mathcal{M}}_g^{G_2,\theta_2}$

Finding $\theta : \Delta \to G = Aut(Y)$ extends both $\theta_1 : \Delta_1 \to G_1$ and $\theta_2 : \Delta_2 \to G_2$ with $Ker(\theta) = Ker(\theta_1) = Ker(\theta_2) = \Gamma_g$

Geometrically

$$
\begin{array}{ccccc}
 & f_{\theta_1} & Y = \mathbb{H}/\Gamma_g & f_{\theta_2} & \\
 & \swarrow & \downarrow & \searrow & \\
\mathbb{H}/\Delta_1 & & f_\theta & & \mathbb{H}/\Delta_2 \\
 & \searrow & \downarrow & \swarrow & \\
 & & \mathbb{H}/\Delta & &
\end{array}
$$

$$G_1 = C_{p_1} \text{ and } G_2 = C_{p_2}$$

Corresponding to groups:

$$
\begin{array}{ccccc}
 & & \Gamma_g & & \\
 & \swarrow & | & \searrow & \\
\Delta_1 & & | & & \Delta_2 \\
 & \searrow & \downarrow & \swarrow & \\
 & & \Delta & &
\end{array}
\qquad
\begin{array}{ccccc}
 & & \langle 1_d \rangle & & \\
 & \swarrow & | & \searrow & \\
C_{p_1} & & | & & C_{p_2} \\
 & \searrow & \downarrow & \swarrow & \\
 & & G & &
\end{array}
$$

We need to look at maximal actions of C_p for isolated strata

Some Results

- Costa-I (2008). $\mathcal{B}_4$ is connected
- Kulkarni (1991). Existence of isolated points in $\mathcal{B}_g$ iff $g = 2$ or 2g+1 a prime ≥ 11
  Isolated points are given by actions $\theta : \Delta(0; p, p, p) \to C_p$, $p = 2g + 1$
  The actions of C_7 in $\mathcal{M}_3$ extend to actions of C_{14} or $PSL(2, 7)$
- Bartolini-I (2009): $\overline{\mathcal{M}}_g^{C_2,\theta}$ and $\overline{\mathcal{M}}_g^{C_3,\theta'}$ belong to the same connected component of $\mathcal{B}_g$.
  All the closed strata induced by actions of C_2 or C_3 intersect the closed stratum formed by surfaces X_g admitting an automorphism of order 2 with quotient Riemann surface of genus highest possible: $\frac{g}{2}$ for even g and $\frac{g+1}{2}$ for odd g.
- Costa-I (2011): $\mathcal{B}_g$ contains isolated strata of dimension 1 iff g+1 is a prime ≥ 11
  The isolated strata are given by actions:
  $\theta_h : \Delta(0; p, p, p, p) \to C_p : \theta_h(x_1) = a, \theta_h(x_2) = a^i, \theta_h(x_3) = a^j$
  $i \neq 1, p-1, j \neq 1, p-1, i, p-i,\ p-1-i-j \neq 1, i, j.$
  These actions are maximal and the strata contain no curve with more symmetry.
- **Branch loci in genera four, seven, thirteen, seventeen, nineteen and fifty-nine are connected**.
  GAP-machinery !!
- Bartolini-Costa-I (2013). These are the only genera with connected branch locus.

Actions given isolated stratum of maximal dimension

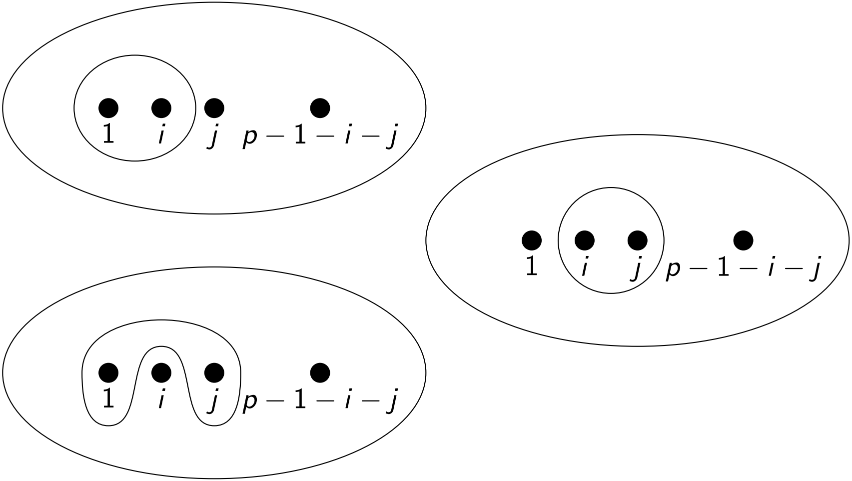
- **g = 60**, action $\theta : \Delta(0; 5^{32}) \to C_5$:
  $\theta(x_1) = \cdots = \theta(x_{19}) = \alpha$, $\theta(x_{20}) = \cdots = \theta(x_{24}) = \alpha^2$, $\theta(x_{25}) = \alpha^3$,
  $\theta(x_{26}) = \cdots = \theta(x_{32}) = \alpha^4$.
- **g = 61**, action $\theta : \Delta(1; 5^{30}) \to C_5$
  $\theta(a) = \theta(b) = 1$, $\theta(x_1) = \cdots = \theta(x_{23}) = \alpha$, $\theta(x_{24}) = \cdots = \theta(x_{28}) = \alpha^2$,
  $\theta(x_{29}) = \alpha^3$, $\theta(x_{30}) = \alpha^4$.
- **g = 63**, action $\theta : \Delta(0; 7^{23}) \to C_7$:
  $\theta(x_1) = \cdots = \theta(x_{14}) = \alpha$, $\theta(x_{15}) = \cdots = \theta(x_{19}) = \alpha^5$, $\theta(x_{20}) = \alpha^4$,
  $\theta(x_{21}) = \cdots = \theta(x_{23}) = \alpha^2$.
- **g = 67**, action $\theta : \Delta(1; 7^{22}) \to C_7$
  $\theta(a) = \theta(b) = 1$, $\theta(x_1) = \cdots = \theta(x_{17}) = \alpha$, $\theta(x_{18}) = \cdots = \theta(x_{20}) = \alpha^6$,
  $\theta(x_{21}) = \alpha^3$, $\theta(x_{22}) = \alpha^4$.
- **g = 71**, action $\theta : \Delta(2; 7^{21}) \to C_7$
  $\theta(a_i) = \theta(b_i) = 1, i = 1, 2$, $\theta(x_1) = \cdots = \theta(x_{13}) = \alpha$,
  $\theta(x_{14}) = \cdots = \theta(x_{16}) = \alpha^2$, $\theta(x_{17}) = \theta(x_{18}) = \alpha^5$, $\theta(x_{19}) = \alpha^3$, $\theta(x_{20}) = \alpha^4$,
  $\theta(x_{21}) = \alpha^6$.

Isolated strata in the complection of branch loci

Costa-I-Parlier (2015): The complections in the Deligne-Munford compactification $\widehat{\mathcal{B}}_g$ of isolated strata of dim 1 given by the monodromies θ_h are isolated.

$(\theta_h : \Delta(0; p, p, p, p) \to C_p : \theta_h(x_1) = a, \theta_h(x_2) = a^i, \theta_h(x_3) = a^j)$

The limit points in $\widehat{\mathcal{B}}_g$ of every such stratum (given by a monodromy θ_h with quotient the sphere with four branch points of order p) is the covering given by f_{θ_h} of the limit point of pinched spheres with a decomposition in two pairs of pants, each pair of pants has as boundary two branch points and a curve surrounding two branch points. As in the next slide.


1
i
j
p − 1 − i − j
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Consider the (hyperbolic) orbifold of genus 0 with two branch points of order p and a cusp. The cyclic p-gonal coverings are given by the monodromies

$\theta : \Delta(0; p, p, \infty) =< y_1, y_2 \mid y_1^p = y_2^p > \to < t >$ where $\theta(y_1) = t^a, \theta(y_2)) = t^b$

Two such maps (t^a, t^b) and $(t^{a'}, t^{b'})$ induce equivalent surfaces iif there exists a c such that $a' \equiv ca \mod (p), \quad b' \equiv cb \mod (p)$. Each equivalence class of monodromies has a representative of type $(1, j)$. Call P_j the covering given by the monodromy of type $(1, j)$.

The limit points of each stratum are

$P_i + P_{-1-\frac{i+1}{j}}, \; P_j + P_{-1-\frac{j+1}{i}}, \; P_{\frac{j}{i}} + P_{p-1-i-j}$

where $2 \leq i \leq \frac{p-1}{2}, i < j \leq p-3, p-1-i-j \notin \{1, i, j, p-1, -i, -j\}$

The limit points for other strata of p-gonal Riemann surfaces with quotient the sphere with four branch points are

$P_1 + P_{p-3}; \; P_1 + P_1, \; P_{p-1} + P_{p-1}; \; P_{p-1} + P_{p-1}, \; P_i + P_i, \; P_{-i} + P_{-i}$ with $2 \leq i \leq \frac{p-1}{2}$ and $P_1 + P_{\frac{p-i-2}{i}}, \; P_i + P_{p-i-2}$ where $2 \leq i \leq \frac{p-1}{2}$

Using elementary number theory, the limit points $P_i + P_{-1-\frac{i+1}{j}}, \; P_j + P_{-1-\frac{j+1}{i}}$,

$P_{\frac{j}{i}} + P_{p-1-i-j}$ do no coincide with limit points of other stratum.

Finally these limit points do not admit any other automorphism.

$\mathcal{B}^{k,Hyp}_{(g,-,1)}$ consits of $\frac{g+2}{2}$ connected components if g even

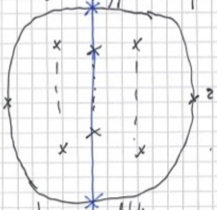
$\frac{g+1}{2}$ connected components if g odd

Consider Y hyperelliptic surface of top type $t=(g,-,1)$; $Y = \mathbb{H}/\Gamma$ Γ S.Gr

Γ index 2 subgr in Δ: $s(\Delta) = (0;+;[2,\overset{g}{\ldots},2],\{(2,2)\})$

$Aut(Y) \leq C_2 \times C_2$ (Bujalance-Etayo-Gamboa-Gromadzki, 1990)

Geometrically, we have the configuration for the action of $Aut(Y)/\langle\varphi\rangle$ (φ hyperelliptic involution)



each arc of the boundary setwise fixed

$2r+s = g+3$

If we have $\theta_r: \Lambda \longrightarrow C_2\times C_2 = \langle a,b\rangle$ $0\leq r\leq [g/2]$

$s(\Lambda) = (0;+;[2^r];\{(2^s)\})$ and monodromies

$\theta_r: \Lambda \longrightarrow C_2\times C_2 = \Lambda/\Gamma$; $s(\theta^{-1}\langle a\rangle) = (0;+;[2^g];\{(2,2)\})$

$\theta_r(x_i) = a$; $\theta_r(e) = a$ / $1d$ according r's parity

$\theta(c_0) = \theta(c_s) = a$; $\theta_r(c_1) = 1d$

Alternating $\theta_r(c_{2j}) = b$; $\theta_r(c_{2j+1}) = ab$

The actions given by θ_r are maximal. They produce $\frac{g}{2}+1 = \frac{g+2}{2}$ connected components for g even and $\frac{g-1}{2}+1 = \frac{g+1}{2}$ connected components for g odd

$\mathcal{B}^{k,Hyp}_{(g,+,1)}$ is connected.

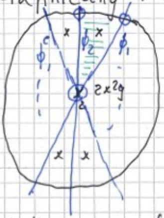
Consider again Y hyper. with top type $t=(g,+,1)$, φ hyper. involution
$Y = \mathbb{H}/\Gamma$ and $Y/\langle\varphi\rangle = \mathbb{H}/\Lambda$ with $s(\Delta) = (0;+;[2^{2g+1}];\{(-)\})$
(a disc with $2g+1$ cone pts

The groups of automorphisms of $Y/\langle\varphi\rangle$ can be dihedral or cyclic

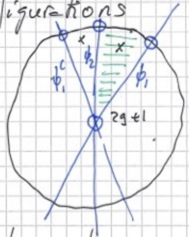
$Aut(Y)$: $C_n \times C_2$, n a proper divisor of $2g+1$
$s(\Lambda) = (0;+;[n,2^r],\{(-)\})$ $r = \frac{2g+1}{n}$
C_{2n} : n a proper divisor of $2g$
$s(\Lambda) = (0;+;[2n,2^r],\{(-)\})$, $r = 2g/n$
D_n : n an even divisor of $4g$
$s(\Lambda) = (0;+;[2^r];\{(n;2^{s-2})\})$; $s = \frac{4g}{n} + 2 - 2r$
$D_{n/2} \times C_2$; n an even divisor of $4g+2$
$s(\Lambda) = (0;+;[2^r];\{(\frac{n}{2},2^s)\})$; $s = \frac{4g+2}{n} + 2 - 2r$

(Bujalance - Etayo - Gamboa - Gromadzki, 1990)

Graphically: Consider configurations

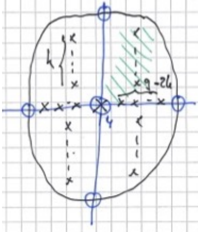


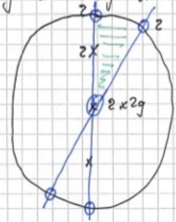
or

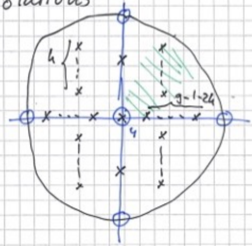


They connect rotations and anticonformal involution of top. type $(0; +; [2, g-2], \{(2,2,2)\})$

The following configurations show actions connecting all the strata induced by anticonformal involutions







THANK YOU