Graph limits and Markov spaces

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What is a limit theory?

$G_1, G_2, \ldots, G_n, \ldots$ more and more similar. Template for large G_n?

Plan

Convergence and limit of finite structures: a bit of history

Left and right convergence, limits

- Dense graph sequences and graphons

- Bounded-degree graph sequences and graphings

What happens inbetween?

- Markov spaces and double measure spaces

Why limits?

Early contructions

- Continuous geometries von Neumann 1936
- Subgraph counts in the limit Erdős-LL-Spencer 1979
- Continuous partition lattices Björner 1986
- Continuous number fields, matroids,... Björner-LL 1987
- Convergence of metric spaces Gromov 1989

Graph Limits

- Scaling limits in statistical physics
- Planar/bounded degree graphs

  Benjamini, Schramm 2001

- Dense graphs

  Borgs, Chayes, LL, T. Sós, Szegedy, Vesztergombi 2003

- First order convergence

  Nesetril, Ossona de Mendez 2010

More limits

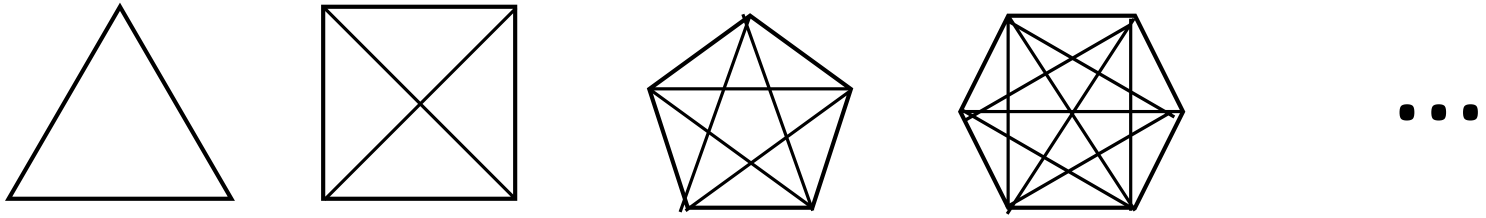
- Limits of partially ordered sets   Janson   2011
- Limits of permutations

  Hoppen, Kohayakawa, Moreira, Ráth and Menezes Sampaio 2011
- Limits of functions on Abelian groups

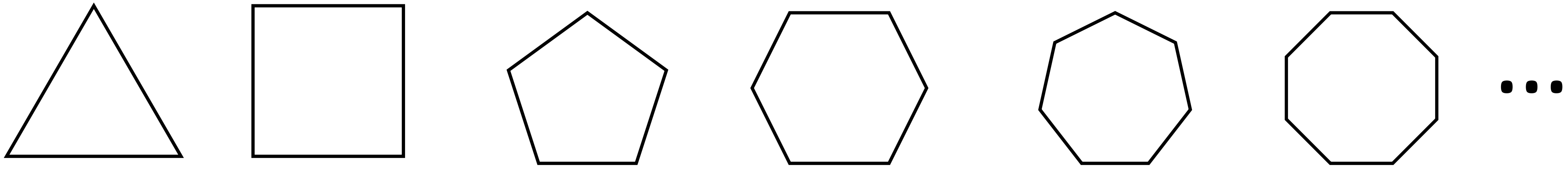
  Szegedy 2012, Green, Tao, Ziegler 2011
- Limits of dense hypergraphs   Elek, Szegedy 2012

„…more and more similar”?

Complete graphs

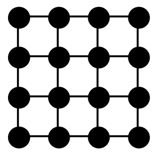
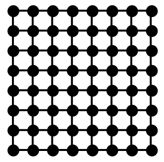
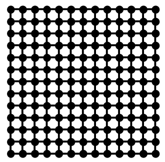


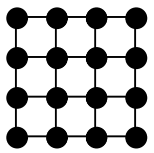
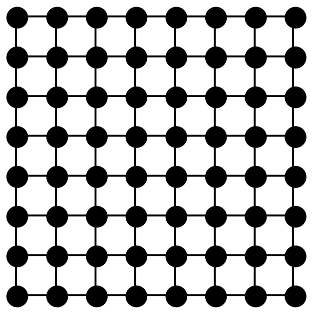
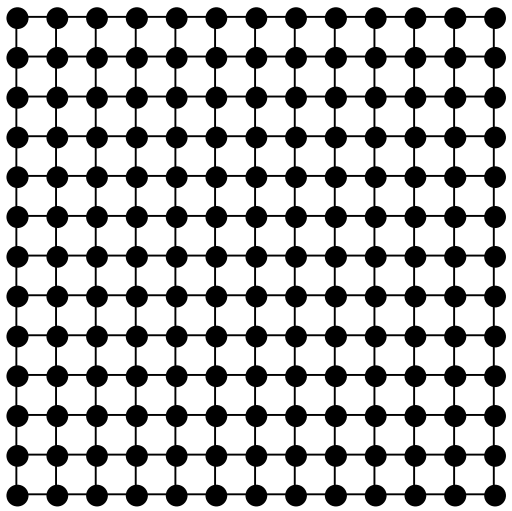
Cycles



„…more and more similar”?

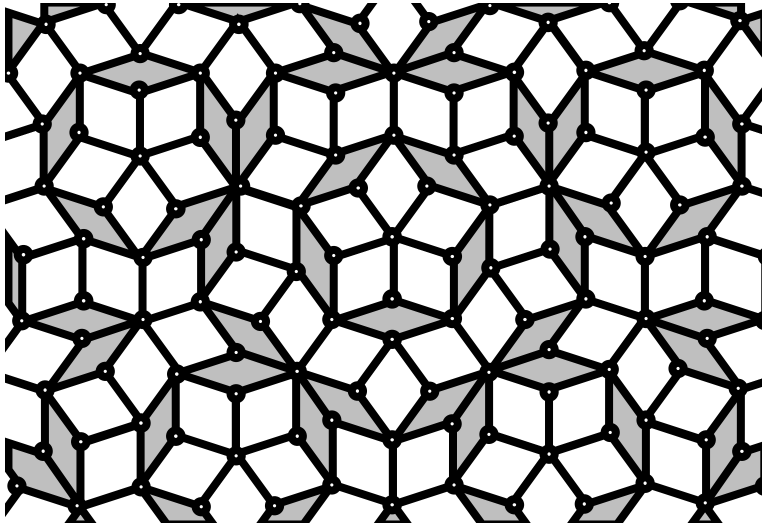
Grids

„…more and more similar”?

Penrose tilings



„...more and more similar”?

Erdős-Rényi random graphs $G(n, 1/2)$ ($n \to \infty$)

What is a limit theory?

$G_1, G_2, \ldots, G_n, \ldots$ more and more similar. Template for large G_n?

notions of convergence

construction
of limit objects

Left and right convergence

$$F \rightarrow \color{red}{G} \rightarrow H$$

subgraph counts

neighborhood statistics

degree distribution

eigenvalues

statistical physics models

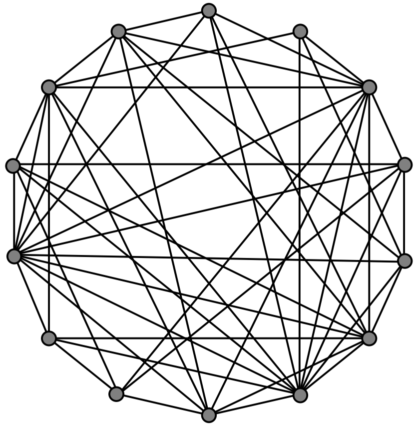
chromatic polynomial

maximum cut

regularity partitions

eigenvectors

Pixel pictures



0	0	1	0	0	1	1	0	0	0	1	0	0	1
0	0	1	0	1	0	1	0	0	0	0	0	1	0
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0	0	1	0	1	0	1	0	1	0	1	1	0	0
0	1	0	1	0	1	1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1	0	1	1	1	0	1
1	1	1	1	1	1	0	1	0	1	1	1	1	0
0	0	1	0	0	1	1	0	1	0	1	0	1	1
0	0	1	1	0	0	0	1	1	1	0	1	0	0
0	0	0	0	0	1	1	0	1	0	1	0	1	0
1	0	1	1	1	1	1	1	0	1	0	1	1	1
0	0	0	1	0	1	1	0	1	0	1	0	1	0
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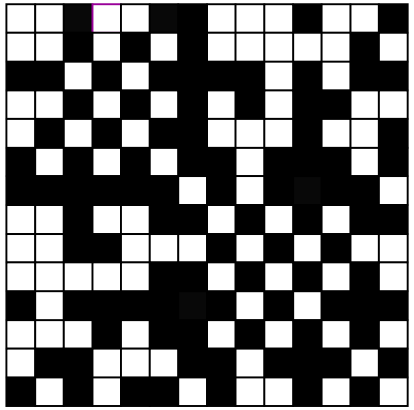
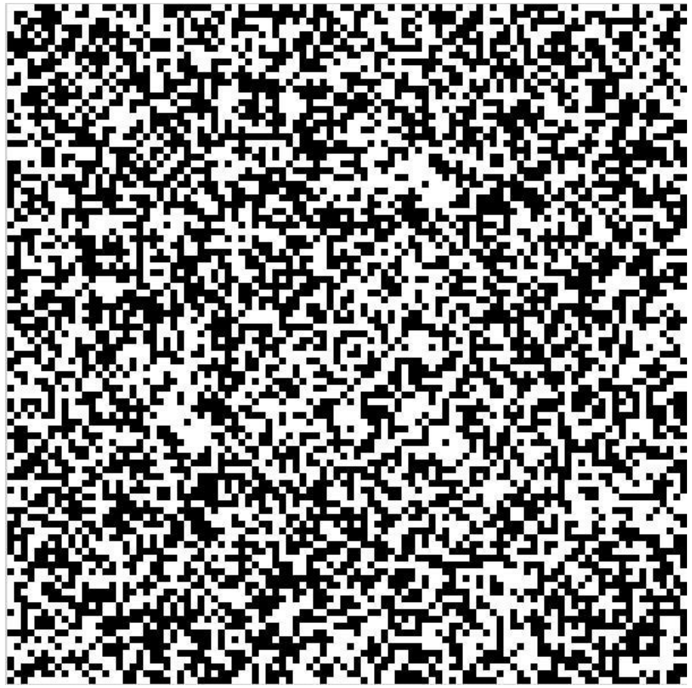
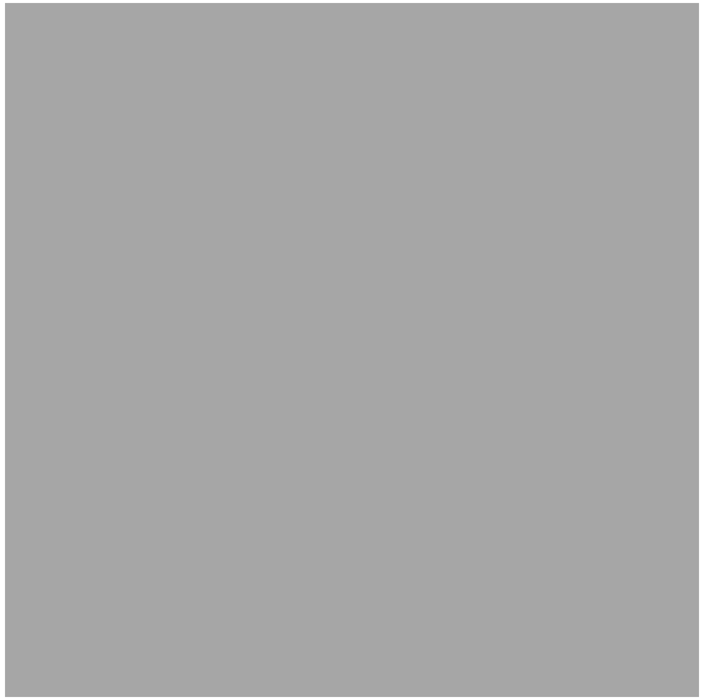


Limits of dense graph sequences



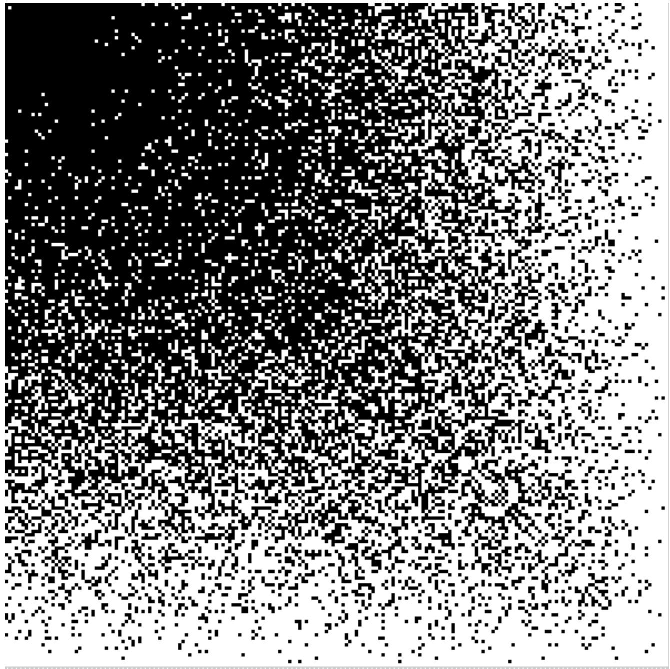
A random graph

with 100 nodes and 2500 edges



1/2

Limits of dense graph sequences



Randomly grown uniform attachment graph with 200 nodes

$1 - \max(x, y)$

Limits of dense graph sequences

Limit objects: graphons

W: $[0,1]^2 \to [0,1]$,
symmetric, measurable

Extending graph theory to graphons

Connectivity, matchings, spectra, automorphisms,…

Dense graphs: left-convergence

Density of F in G = prob that random map $V(F) \to V(G)$ preserves edges: $t(F,G) = \dfrac{\#\text{homomorphisms } F \to G}{|V(G)|^{|V(F)|}}$

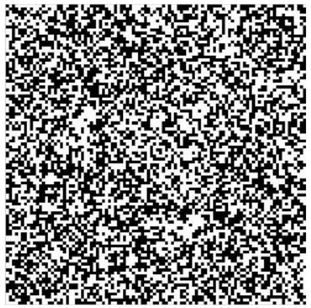
$(G_1, G_2, \ldots)$ is left-convergent, if $(t(F,G_1), t(F,G_2), \ldots)$ is convergent $\forall F$.

Dense graphs: left-convergence to the limit

Density of graph *F* in graphon *W*: $t(F,W) = \int_{[0,1]^{V(F)}} \prod_{ij \in E(F)} W(x_i, x_j)\, dx$

$G_n \to W$, if $t(F,G_n) \to t(F,W)$ for $\forall F$.

Erdős-Rényi random graph: $G(n,\frac{1}{2}) \to W \equiv \frac{1}{2}$



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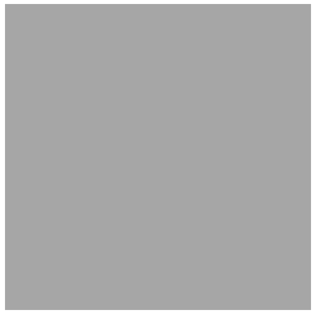


Dense graphs (main facts)

For every left-convergent graph sequence (G_n)
there is a graphon W such that $G_n \to W$.

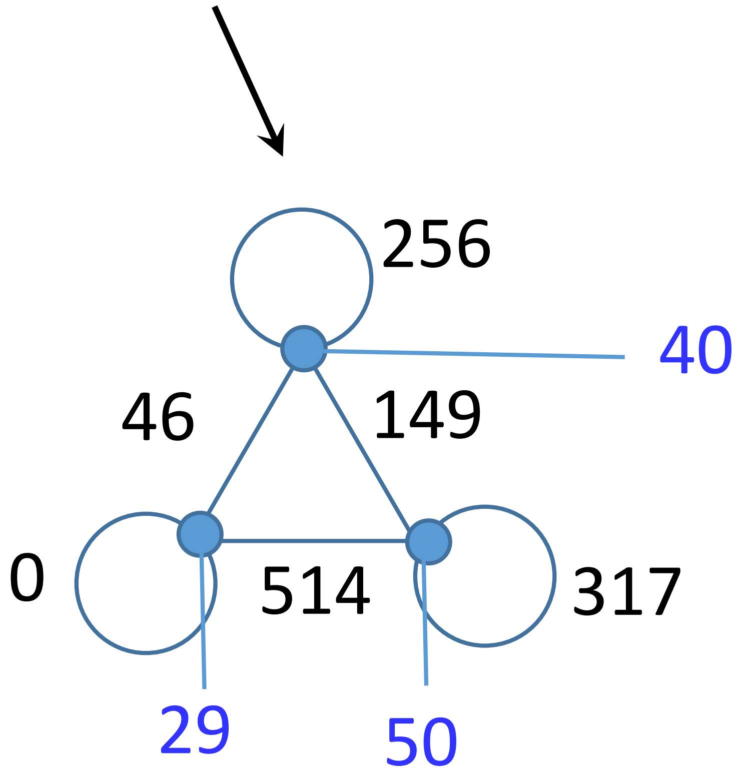
For every graphon W
there is a graph sequence (G_n) such that $G_n \to W$.

soficity

The limit graphon is essentially unique
(up to measure preserving transformations
and changes of measure 0).

Dense graphs: right-convergence

$F \rightarrow$ G $\rightarrow H$



$G/\mathcal{P}$

$Q_q(G) = \{G/\mathcal{P} : \mathcal{P}$ a q-partition of $V(G)\}$

$G_1, G_2, \ldots$ is (left)-convergent iff

$\forall q\ Q_q(G_n)$ (normalized) is convergent

in Hausdorff distance

Local convergence of bounded-degree graphs

$$\frac{\#\text{homomorphisms } F \to G_n}{|V(G_n)|}$$ converges $\forall$ connected F

Limit objects: involution-invariant distributions on countable rooted graphs

Benjamini - Schramm

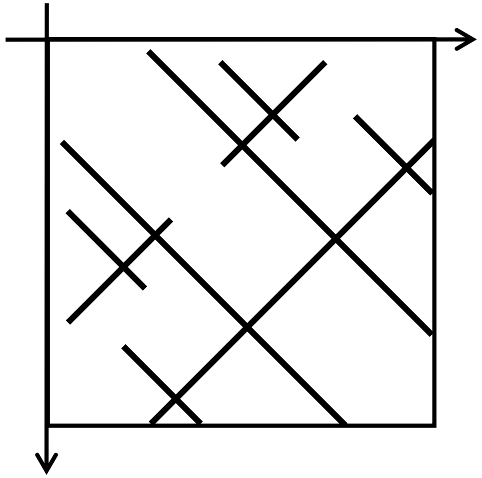
Soficity:
Aldous-Lyons Conjecture
Gromov Problem:
$\forall$countable group is sofic

Local convergence of bounded-degree graphs

Limit obejct: graphing ≡ Borel graph on [0,1] with bounded degree

with "double counting" condition

$$\int_A \deg(x,B)\,dx = \int_B \deg(x,A)\,dx$$



Extending graph theory to graphings

Matchings, flows, expansion, edge-coloring,…

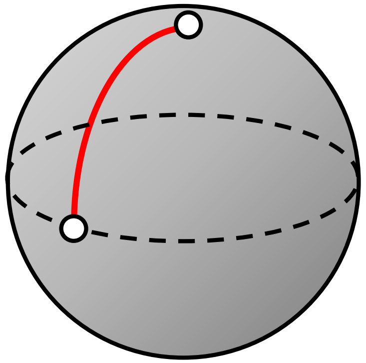
Intermediate densities

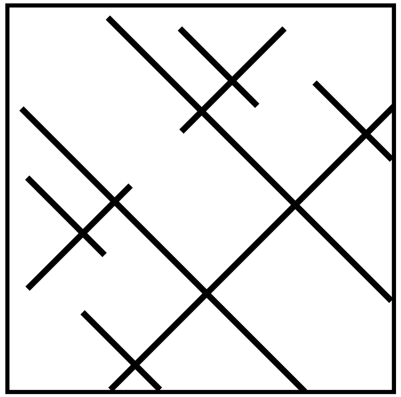
Current focus of research

Continuum node set

Borgs, Chayes, Cohn, Zhao,
Frenkel, Backhausz, Kunszenti-Kovács, LL, Szegedy

random node + random edge → symm probability measure on $[0,1]^2$





Markov spaces

Markov space: $(J, \mathcal{A}, \eta)$, where $(J,\mathcal{A})$ is a (Borel) sigma-algebra, η is a (symmetric) probability measure on $(J^2,\mathcal{A}^2)$ with equal marginals;

Stationary distribution: marginals $\pi(X)= \eta(X \times J)= \eta(J \times X)$

For every Markov space there is a reversible Markov chain with stationary distribution π and ergodic measure η, and vice versa.

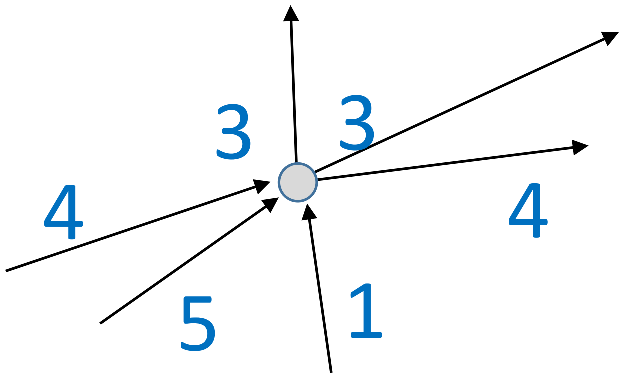
Double measure spaces

$(J, \mathcal{A}, \eta, \lambda)$, where $(J, \mathcal{A}, \lambda)$ is a probability space, and
$(J, \mathcal{A}, \eta)$ is a Markov space.

$\pi(X) = \lambda(X)$ generalizes regular graphs

Extending graph theory to Markov spaces
Flows, expansion, spectra, random walks,...

Flow theory: circulation



$$\sum_j f(ij) = \sum_j f(ji)$$

flow condition

circulation: measure on $[0,1]^2$ with equal marginal

Hoffman Circulation Theorem:

For $\forall$ two measures φ, ψ on $[0,1]^2$, $\exists$ circulation α such that $\varphi \le \alpha \le \psi$, iff $\varphi \le \psi$ and $\varphi(X \times X^c) \le \psi(X^c \times X)$ for every $X \subseteq [0,1]$.

LL

Convergence to Markov spaces

left-convergence

right-convergence

Lyons
Borgs, Chayes, Cohn, Zhao
Frenkel

Kunszenti-Kovács, LL, Szegedy
Backhausz, Szegedy

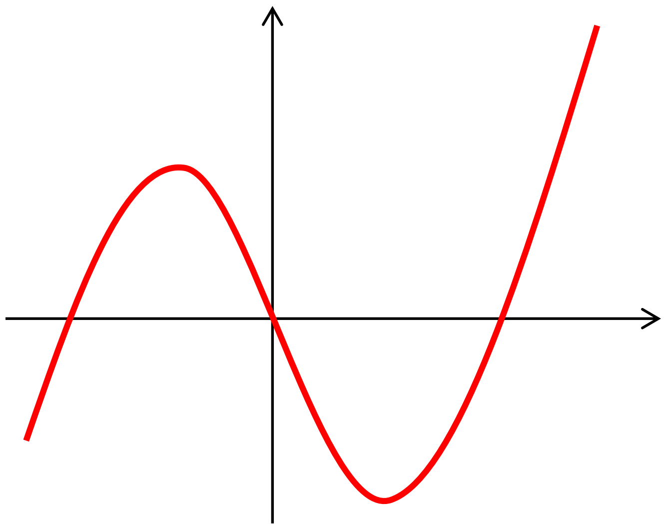
How to define subgraph densities in Markov spaces?

What are graph limits good for?

- Existence of optima
- Large deviation theory for random graphs
- Templates for solutions of extremal graph problems (finite forcing)
- Local extremal graph theory

Existence of optima

Minimize x^3-6x over $x \geq 0$.



minimum is not attained in rationals

⇒ real numbers are useful

Existence of optima

Minimize density of 4-cycles in a graph with edge-density ½.

always >1/16,
arbitrarily close for random graphs

minimum is not attained among graphs $\Rightarrow$ graph limits are useful

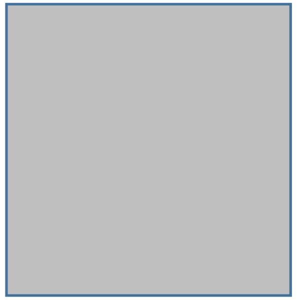
Minimum is attained for constant ½ graphon only.

Finitely forcible graphons

Graphon W is finitely forcible: $\exists\, F_1,\dots,F_m,\ \alpha_1,\dots,\alpha_m$:

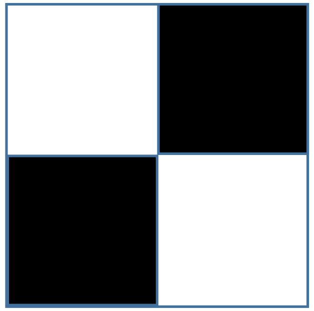
$$\left.\begin{aligned} t(F_1,W) &= \alpha_1 \\ &\vdots \\ t(F_m,W) &= \alpha_m \end{aligned}\right\} \Rightarrow W \text{ is determined (up to...)}$$

Finitely forcible graphons



constant p functions — Chung-Graham-Wilson 1989

$$t(K_2,W)=p,\ \ t(C_4,W)=p^4$$

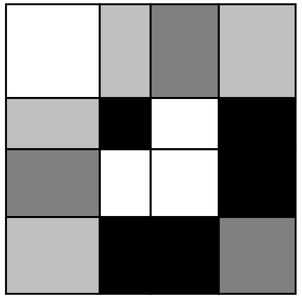


complete bipartite graphs — Mantel - Turán

$$t(K_2,W)=\frac{1}{2},\ \ t(K_3,W)=0$$

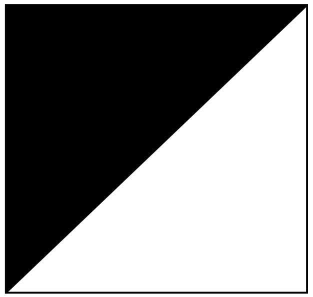
Finitely forcible graphons $\approx$ templates for optimal graphs in extremal graph theory ?

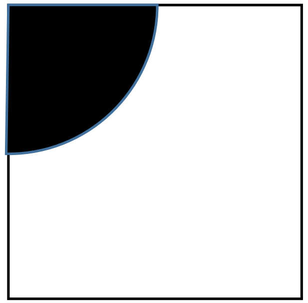
Finitely forcible graphons



stepfunctions

LL-T. Sós 2008





LL-Szegedy 2011

Finitely forcible graphons

Finitely forcible: Baire category I

LL- Szegedy 2011

Not finitely forcible: Baire category II

$$W(x,y) = \begin{cases} \dfrac{x+y}{2} \\ xy \end{cases}$$

Finitely forcible graphons

Several conjectures

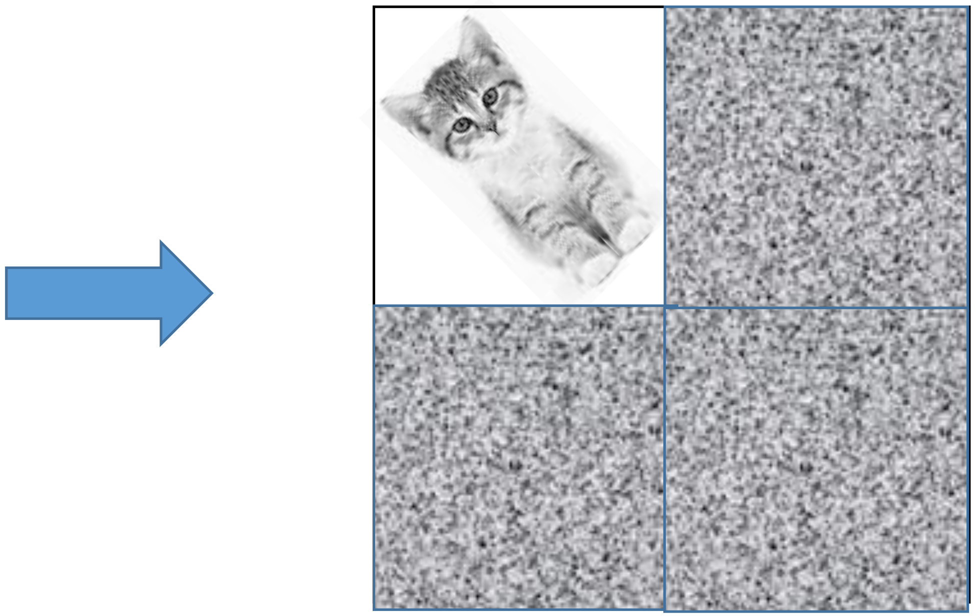
Extremal graph $\Rightarrow$ Finitely forcible $\Rightarrow$ Nice properties (polynomial size Szemerédi partitions, ...)

Several conjectures X

Finitely forcible graphons

Kral, Cooper, Glebov, Grzesik, Kaiser, Klimosova, L.M.Lovász, Martins, Noel, Sosnovec 2013-2020

arbitrary graphon



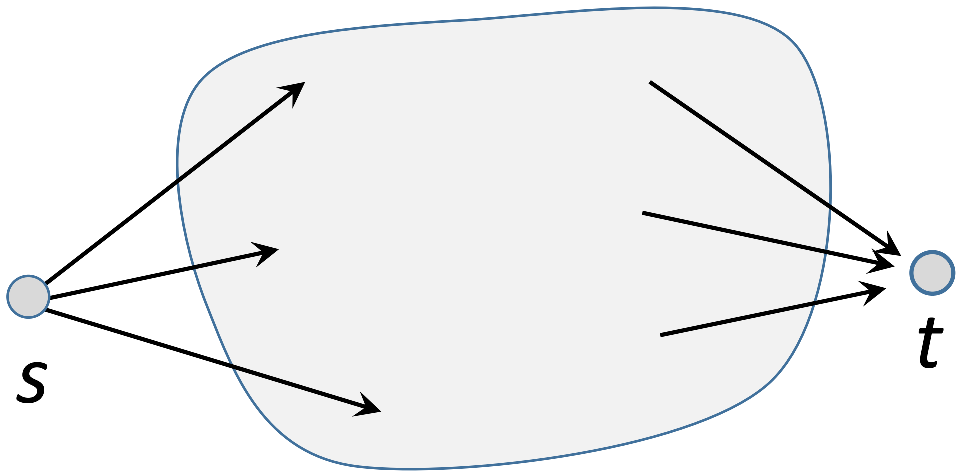
finitely forcible

Thank you for your attention!

Dense graphs (further things to define)

- distance of graphs/graphons in which convergence ⇔ Cauchy
- metric space of graphons (compact)
- regularity partitions of graphons → algorithmic theory of graphons
- spectra of graphons
- extremal graphon theory…

Flows



Flow of value ω: measure φ on $[0,1]^2$

s.t.

$$\varphi^1 - \varphi^2 = \omega\,(\delta_t - \delta_s)$$

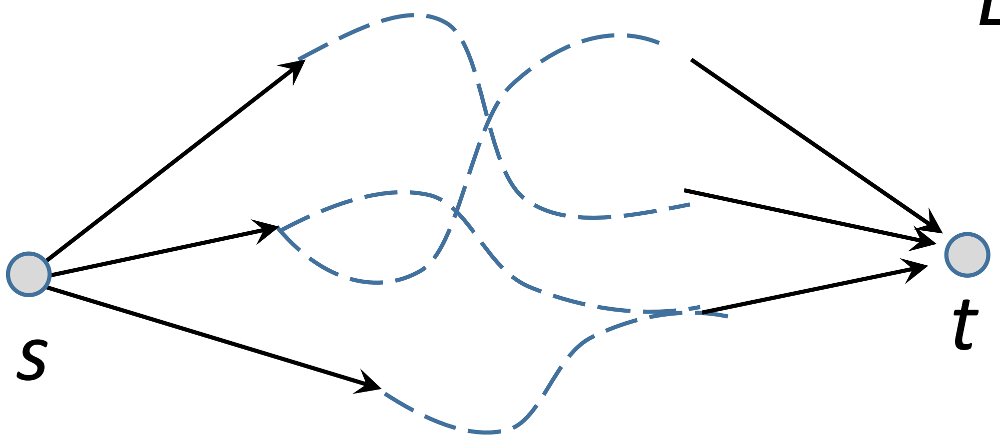
value:

$$\omega(f) = \sum_j f(sj) - \sum_j f(js)$$

$$= \sum_j f(jt) - \sum_j f(tj)$$

Max-Flow-Min-Cut etc. generalizes rather straightforwardly

Decomposition of flows into paths



$B = \{(s, x_1, \ldots x_r, t) : x_i \in [0,1]\}$: $s-t$ paths

τ : measure on B

$\hat{\tau}(S) = \int_B |S \cap E(P)| \, d\tau(P)$:

measure on $[0,1]^2$

For every acyclic s-t flow $\varphi \geq 0$ there is a τ with $\hat{\tau} = \varphi$.

no circulation α with 0≤ α ≤ φ