# Numerical computation of the complex zeros of Bessel and Hankel functions

Diego Ruiz Antolín

in collaboration with Amparo Gil and Javier Segura

UNIVERSIDAD DE CANTABRIA

## Type of cylinder functions considered

- Bessel functions of first kind $J_\nu(z)$
- Bessel functions of second kind $Y_\nu(z)$
- Hankel functions $H^{(1)}(z)$ and $H^{(2)}(z)$
- General combinations of Bessel and Hankel functions

$$\alpha J_\nu(z) + \beta Y_\nu(z)$$

$$\alpha H^{(1)}(z) + \beta H^{(2)}(z)$$

## Historical background


- J. Segura. Reliable computation of the zeros of solutions of second order linear ODEs using a fourth order method. SIAM J. Numer. Anal., 48(2):452–469, **2010**.
- J. Segura. Computing the complex zeros of special functions. Numer. Math., 124(4):723–752, **2013**.
- A. Gil and J. Segura. On the complex zeros of airy and bessel functions and those of their derivatives. Anal. Appl., 12(5):573–561, **2014**.

## Key elements of the algorithm

Given a function $y(x)$ defined to be a solution of a second order linear differential equation

- Qualitative analysis of the approximate Liouville-Green Stokes lines (SLs) and anti-Stokes lines (ASLs) for the differential equation.
- The structure of the exact zeros will follow very closely the ASLs.
- Combine this analysis with the application of a fixed point method $\omega_{n+1} = T(\omega_n)$ (of order four) and carefully selected step functions $H^{\pm}(\omega)$ (for displacement between zeros).

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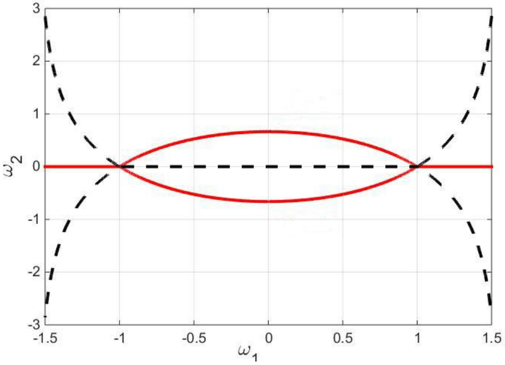
## Outline of the algorithm

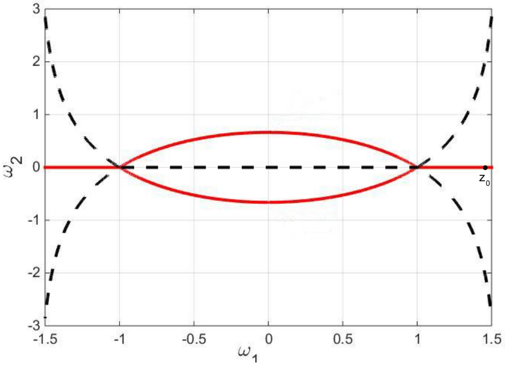
1. Divide the complex plain in disjoint domains separated by the principal SLs and ASLs and compute in each domain.
2. In each domain, start away from the principal SLs and close to the principal ASLs and/or a singularity (if any) and iterate with $T(\omega)$ until a first zero is found. If a value outside the domain is reached, stop the search in that domain.
3. Alternate between the fixed point method $T(\omega)$ and the displacement function $H^{\pm}(\omega)$ in the direction of approach to the principal SLs.

## Graphic example



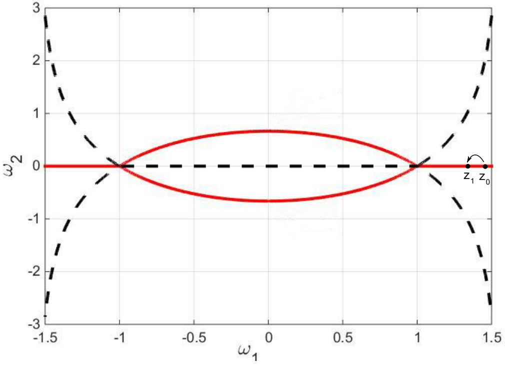

Principal Stokes and anti-Stokes lines of a general function $Y_\nu(z)$ for positive $\nu$.

## Graphic example



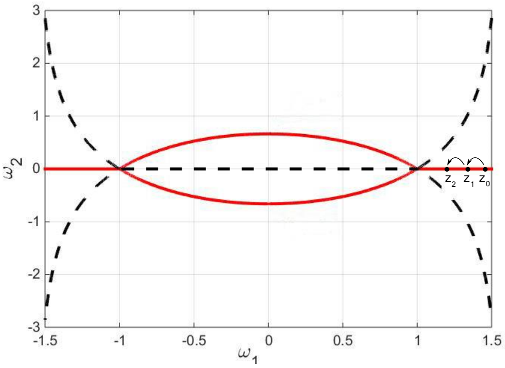

We start at a point on the positive real axis or above it and away from the Stokes lines.

## Graphic example




We iterate the process using the fixed point method $T(\omega)$ and the step function $H^{\pm}(\omega)$ conveniently to find the subsequent zeros.

# Graphic example

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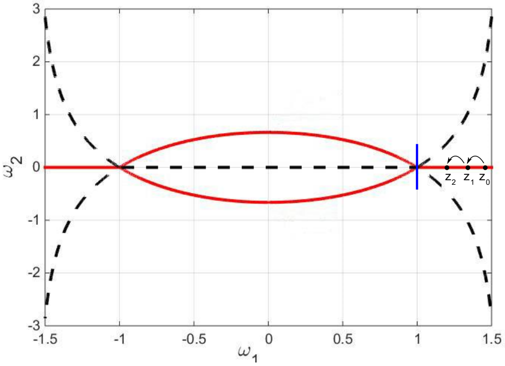
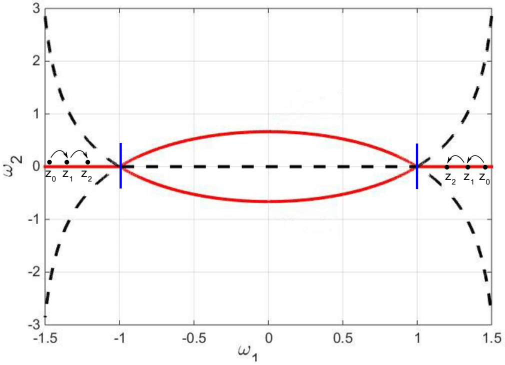


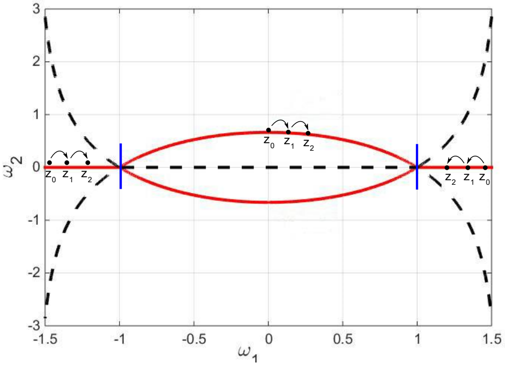

The root finding method stops when a principal Stokes line is reached.

## Graphic example



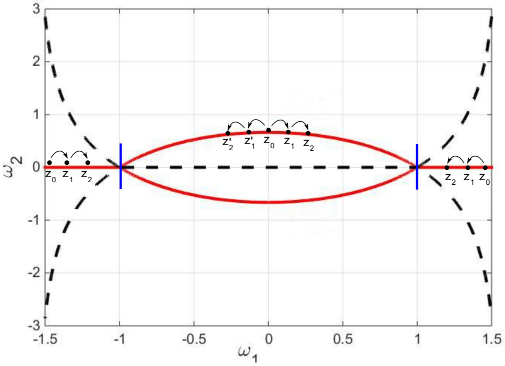

We start again the process above the negative real axis until another principal Stokes line is reached.

## Graphic example



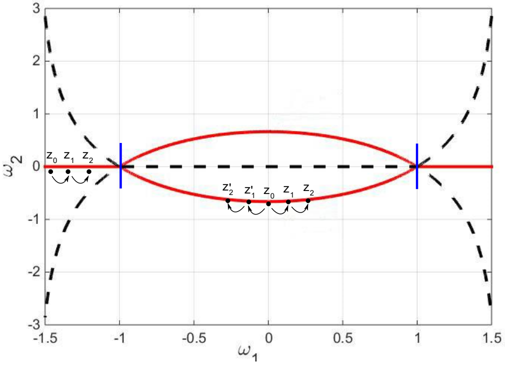

We proceed now to do the same process along the eye-shaped anti-Stokes line.

## Graphic example




When a Stokes line is reached, we go back and proceed again in the other direction.

## Graphic example




We repeat the process for the negative imaginary part of the complex plane, obtaining the rest of the zeros.

## The MATLAB routine

```
[nz,zer,ierr]=zercylg(a,icho,alpha,beta,Lx,Ly)
```

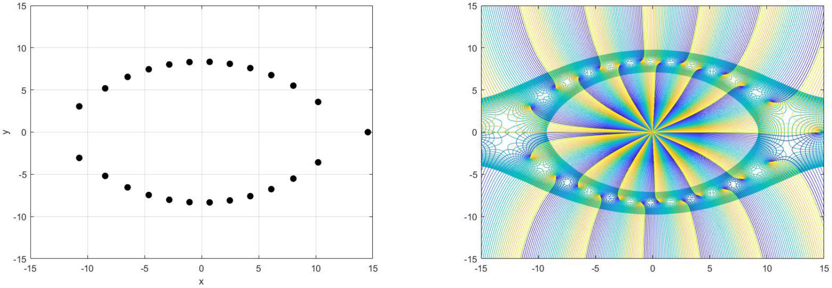
Input

- a : order of the function
- icho : choice of the function
- alpha, beta : parameters for the combinations
- Lx, Ly : to indicate the size of the rectangle $[-Lx, Lx] \times [-Ly, Ly]$

Output

- nz : number of zeros found
- zer : array with the zeros found
- ierr : error flag

## Some examples of the performance




zercylg(12.3,2,1,1,15,15)

Left: plot of the zeros of $Y_{12,3}(z)$.
Right: contour plot of the modulus and phase of the function.

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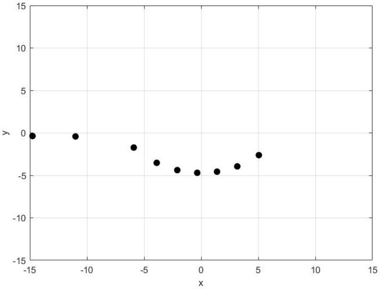
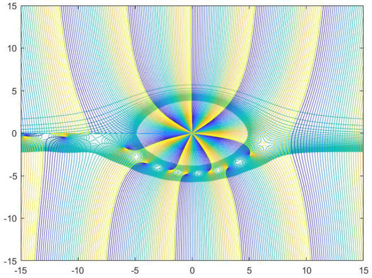
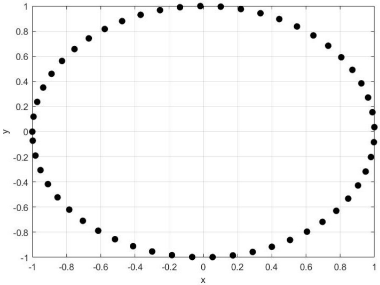


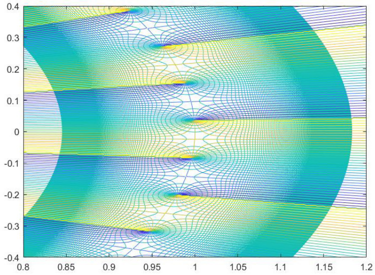




`zercylg(-7.1,3,1,1,15,15)`

Left: plot of the zeros of $H^{(1)}_{-7,1}(z)$.
Right: contour plot of the modulus and phase of the function.

## Some examples of the performance




```
zercylg(26.3,5,complex(10.5,1.0),complex(1.0,10.5),1,1)
```

Left: plot of the zeros of $\alpha J_{26,3}(z) + \beta Y_{26,3}(z)$.
Right: contour plot of the modulus and phase of the function in the box $[0{,}8,\ 1{,}2] \times [-0{,}4,\ 0{,}4]$

# Testing the efficiency and accuracy

- Tabulated values from other sources.
- Maple approximations in extended precision.
- Comparison with the ZEBEC algorithm.

P. Kravanja, O. Ragos, M.N. Vrahatis, and F.A. Zafiropoulos. ZEBEC: A mathematical software package for computing simple zeros of Bessel functions of real order and complex argument. Comput. Phys. Commun.,113:220–238, 1998.

A. Gil, D. Ruiz-Antolín, J. Segura., An algorithm for computing the complex zeros of Bessel and Hankel functions *In progress*.

Thank you for your time