Local asymptotics and unique continuation from boundary points for fractional equations

Joint work with A. De Luca and S. Vita

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Fractional elliptic equations

$$\begin{cases} (-\Delta)^s u = h\, u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \qquad (E_s)$$

where $s \in (0,1)$ and $\Omega \subset \mathbb{R}^N$ is open and bounded with $N \geq 2$
$0 \in \partial\Omega$ and $\partial\Omega$ is of class $C^{1,1}$ in a neighbourhood of 0

The fractional Laplacian $(-\Delta)^s$ of a function $u : \mathbb{R}^N \to \mathbb{R}$ is defined as

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \widehat{u}(\xi),$$

where $\widehat{u}$ is the Fourier transform of u, i.e.

$$\widehat{u}(\xi) = \mathcal{F}u(\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} u(x)\, dx\,.$$

Problem: strong unique continuation property and local asymptotics of solutions (blow-up analysis and quantization of the possible vanishing orders) at $0 \in \partial\Omega$.

Weak formulation of (E_s)

Let $\mathcal{D}^{s,2}(\mathbb{R}^N)$ be the completion of $C_c^\infty(\mathbb{R}^N)$ w.r.t. the norm induced by the scalar product

$$(u, v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s}\, \overline{\widehat{v}(\xi)}\, \widehat{u}(\xi)\, d\xi.$$

$(-\Delta)^s$ can be extended to a bounded linear operator from $\mathcal{D}^{s,2}(\mathbb{R}^N)$ to its dual $(\mathcal{D}^{s,2}(\mathbb{R}^N))^*$, the Riesz isomorphism of $\mathcal{D}^{s,2}(\mathbb{R}^N)$:

$${}_{(\mathcal{D}^{s,2}(\mathbb{R}^N))^*}\langle(-\Delta)^s u, v\rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = (u, v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)}.$$

Definition

A weak solution to (E_s) is a function $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ s.t. $u(x) = 0$ for a.e. $x \in \mathbb{R}^N \setminus \Omega$ and

$$(u, \varphi)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \int_\Omega h(x)u(x)\varphi(x)\, dx \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Unique continuation

Let $\mathcal{F}$ be a family of functions (e.g. the family of solutions of some equation).

Definition

- $\mathcal{F}$ enjoys the **strong unique continuation property (SUCP)** if no function in $\mathcal{F}$, besides possibly the zero function, has a zero of infinite order.

Unique continuation

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Definition

- $\mathcal{F}$ enjoys the **strong unique continuation property (SUCP)** if no function in $\mathcal{F}$, besides possibly the zero function, has a zero of infinite order.
- $\mathcal{F}$ enjoys the **weak unique continuation property (WUCP)** if no function in $\mathcal{F}$, besides possibly the zero function, vanishes on an open set.

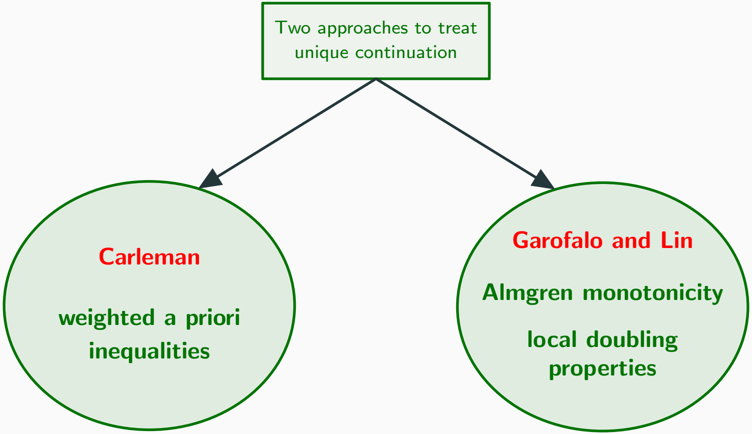
Unique continuation for second order elliptic equations

$\mathcal{F}$ = set of solutions to

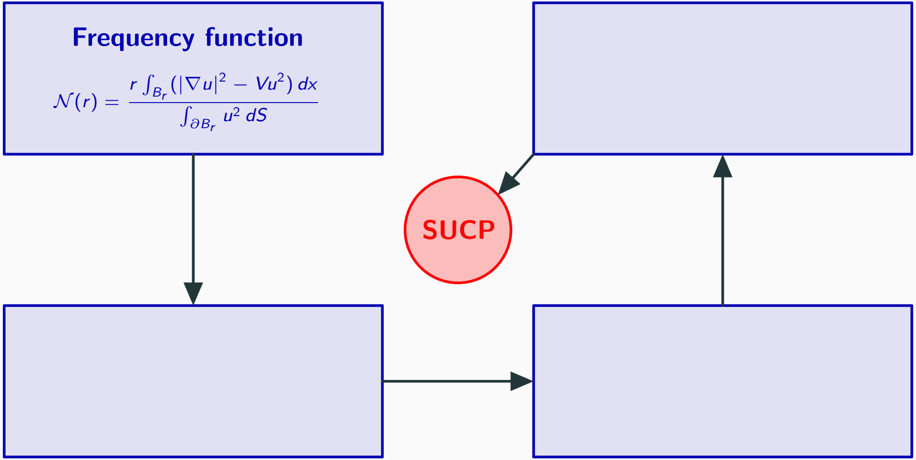
$$-\Delta u = V\, u \quad \text{in } \Omega \subset \mathbb{R}^N$$

- **Carleman (1939)**: $n = 2$, V bounded
- **Aronszajn (1957)**: $n \geq 3$
- **Jerison-Kenig (1985)**: $V \in L^{N/2}_{\mathrm{loc}}$
- **Garofalo-Lin (1986)**: 2nd order elliptic operators with variable coefficients; admits the case $V(x) = \frac{1}{|x|^m}$ with $0 \leq m \leq 2$ (SUCP fails with $m > 2$)
- **Fabes-Garofalo-Lin (1990)**: V in some Kato class
- **Wolff (1992)**: WUCP for solutions to $|\Delta u| \leq V|u| + W|\nabla u|$, with $V \in L^{N/2}_{\mathrm{loc}}$, $W \in L^{N}_{\mathrm{loc}}$
- **Koch-Tataru (2001)**: more general elliptic operators

Unique continuation for second order elliptic equations



SUCP from interior points (e.g. $0 \in \Omega$) via monotonicity for $-\Delta u = Vu$



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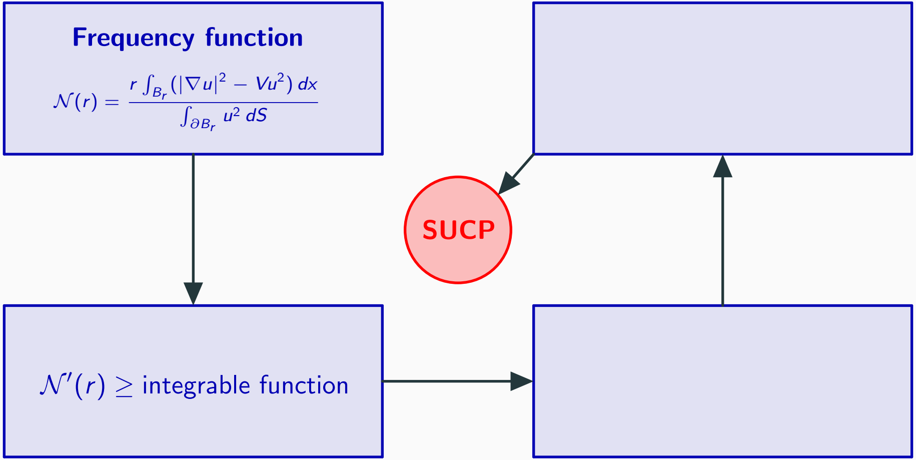

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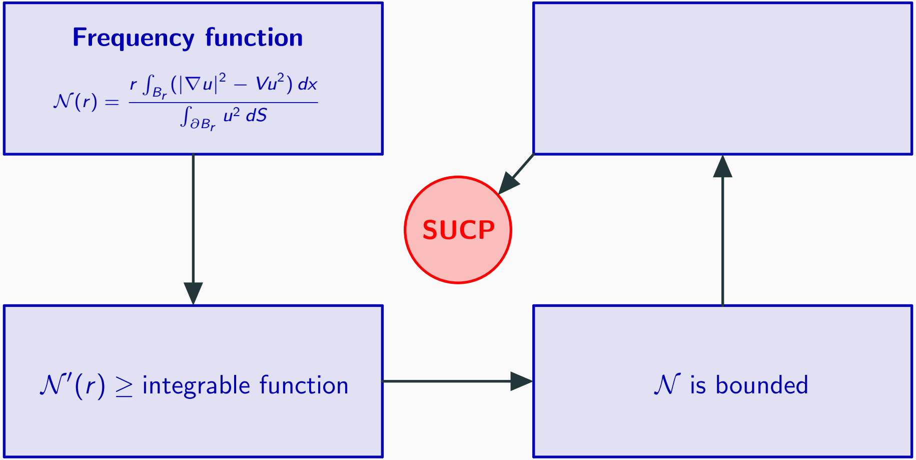

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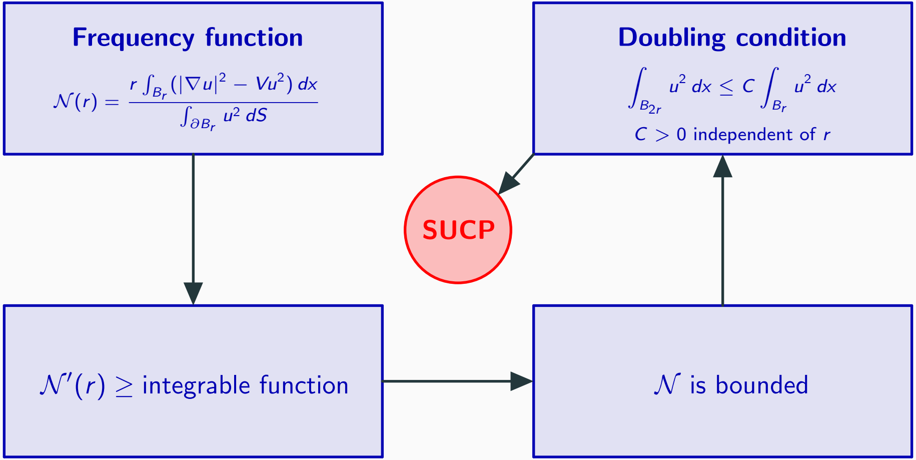

To differentiate $\mathcal{N}$ ...

one integrates the Rellich–Nečas identity

$$\operatorname{div}\left(|\nabla u|^2 x - 2(\nabla u \cdot x)\nabla u\right) = (N-2)|\nabla u|^2 - 2(\nabla u \cdot x)\Delta u$$

on balls $B_r \subset \Omega$, obtaining a Pohozaev-type identity

$$-\frac{N-2}{2}\int_{B_r} |\nabla u|^2 \, dx + \frac{r}{2}\int_{\partial B_r} |\nabla u|^2 \, dS = r\int_{\partial B_r} \left|\frac{\partial u}{\partial \nu}\right|^2 \, dS + \int_{B_r} Vu(\nabla u \cdot x)\, dx.$$

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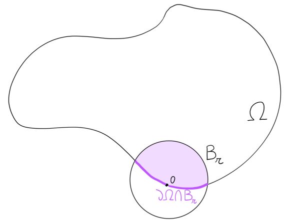
But

- this requires some regularity for u (e.g. $u \in H^2$)
- if $0 \in \partial\Omega$ $\rightsquigarrow$ **loss of regularity**
  interference with the geometry of the domain

$\downarrow$

extra terms arising in the integration by parts
and appearing the rest of $\mathcal{N}'$.



Unique continuation from boundary points

- **Adolfsson-Escauriaza-Kenig (1995), Adolfsson-Escauriaza (1997), Kukavica-Nyström (1998), Tao–Zhang (2008), F.-Ferrero (2013)**: under homogeneous Dirichlet conditions
- **Tao–Zhang (2005), Dipierro-F.-Valdinoci (2020)**: under Neumann type conditions
- **Fall-F.-Ferrero-Niang (2019)**: unique continuation from Dirichlet-Neumann junctions for planar mixed boundary value problems
- **De Luca-F. (2021)**: unique continuation from the edge of a crack.

Unique continuation for fractional Schrödinger equations

- **Fall-F. (2014)**: SUCP and UCP from sets of positive measure for

$$(-\Delta)^s u(x) - \frac{\lambda}{|x|^{2s}} u(x) = h(x)u(x) + f(x, u(x)) \quad \text{with } s \in (0,1)$$

  via **frequency function methods** for the Caffarelli-Silvestre extension;
- **Fall-F. (2015)**: analogous results for relativistic Schrödinger operators;
- **Rüland (2015)**: SUCP for fractional Laplacians with power $s \in (0,1)$ in presence of rough potentials, via Carleman inequalities for the Caffarelli-Silvestre extension;
- **Yu (2017)**: fractional operators with variable coefficients.
- **Yang (2013), Seo (2014-2015), F.-Ferrero (2020), García-Ferrero-Rüland (2019)**: higher order ($s > 1$) fractional equations.

Caffarelli-Silvestre extension

$\mathbb{R}^{N+1}_+ = \{z = (x,t) : x \in \mathbb{R}^N,\ t > 0\}$

$\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) :=$ completion of $C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ w.r.t. the norm

$$\|w\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})} = \left(\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla w(x,t)|^2 dx\, dt\right)^{1/2}$$

- $\exists$ a trace map $\quad \mathrm{Tr} : \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \to \mathcal{D}^{s,2}(\mathbb{R}^N)$
- $\forall u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \quad \exists!\ \mathcal{H}(u) \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ weakly solving

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla \mathcal{H}(u)) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \mathrm{Tr}\, \mathcal{H}(u) = u & \text{on } \partial\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times \{0\}. \end{cases}$$

Caffarelli-Silvestre extension

Caffarelli and Silvestre (2007)

$$-\lim_{t\to 0^+} t^{1-2s}\frac{\partial \mathcal{H}(u)}{\partial t}(x,t) = \kappa_s(-\Delta)^s u(x) \quad \text{in } (\mathcal{D}^{s,2}(\mathbb{R}^N))^\star \qquad \text{where } \kappa_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)} > 0.$$

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$$\Downarrow$$

$$u \text{ solves } (-\Delta)^s u = hu \text{ in } \Omega \iff U = \mathcal{H}(u) \text{ solves } \begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ U(x,0) = u & \text{in } \mathbb{R}^N, \\ -\lim_{t\to 0^+} t^{1-2s}\frac{\partial U}{\partial t}(x,t) = \kappa_s h(x)u(x) & \text{in } \Omega, \end{cases}$$

in a weak sense,

Caffarelli-Silvestre extension

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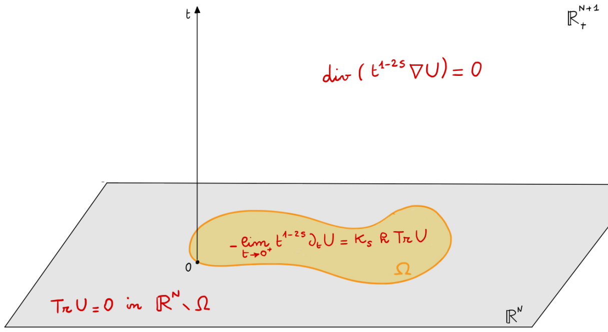
$$u \text{ solves } (-\Delta)^s u = hu \text{ in } \Omega \;\Leftrightarrow\; U = \mathcal{H}(u) \text{ solves } \begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ U(x,0) = u & \text{in } \mathbb{R}^N, \\ -\lim_{t\to 0^+} t^{1-2s}\frac{\partial U}{\partial t}(x,t) = \kappa_s h(x)u(x) & \text{in } \Omega, \end{cases}$$

in a weak sense, i.e. $\operatorname{Tr} U = u$ and $\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla U\cdot\nabla\varphi\,dt\,dx = \kappa_s\int_\Omega hu\operatorname{Tr}\varphi\,dx$

for all $\varphi\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s})$ s.t. $\operatorname{supp}(\operatorname{Tr}\varphi)\subset\Omega$.

Caffarelli-Silvestre extension



We are dealing with a problem with mixed boundary conditions!

Monotonicity formula around $0 \in \partial\Omega$ for the extended problem

Additional difficulties in the development of a monotonicity argument around points located at Dirichlet-Neumann junctions:

interference with the geometry of the domain

lack of regularity at Dirichlet-Neumann junctions

We straight $\partial\Omega$ through a diffeomorphic deformation.

Double approximation procedure:

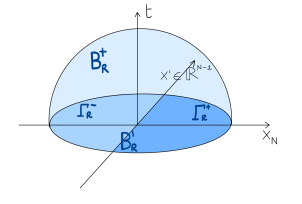
- we approximate the potential h with potentials vanishing near the boundary;
- we approximate the Dirichlet N dimensional region with smooth $(N+1)$-sets with straight vertical boundary.

A diffeomorphism to straighten the boundary

Inspired by **[Adolfsson-Escauriaza (1997)]** we construct $F\colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$, which is a **diffeomorphism of class $C^{1,1}$** from B_R to $\mathcal{U} = F(B_R)$ for some $\mathcal{U}$ open neighbourhood of 0, s.t. $F(x', 0, 0) = (x', g(x'), 0)$ where $B'_R \cap \partial\Omega = \{(x', x_N) \in B'_R : x_N = g(x')\}$

$W = U \circ F$ is solution to

$$\begin{cases} -\mathrm{div}\left(t^{1-2s} A \nabla W\right) = 0 & \text{in } B_R^+, \\ \lim_{t\to 0^+}\left(t^{1-2s} A \nabla W \cdot \nu\right) = \kappa_s \tilde{h}\, \mathrm{Tr}\, W & \text{in } \Gamma_R^-, \\ W = 0 & \text{in } \Gamma_R^+, \end{cases}$$



where $\nu = (0, 0, \dots, 0, -1)$, A is an $(N+1) \times (N+1)$ variable coefficient matrix (not depending on t) (related to the Jacobian matrix of F), and

$$\tilde{h}(y) = \det J_F(y', y_N, 0) h(F(y, 0)), \quad y \in \Gamma_R^-.$$

A diffeomorphism to straighten the boundary

Crucial feature of the matrix A

$$A(y) = \left(\begin{array}{c|c} D(y) & 0 \\ \hline 0 & 1 + O(|y'|^2) + O(y_N) \end{array}\right)$$

where

$$D(y', y_N) = \left(\begin{array}{c|c} \mathrm{Id}_{N-1} + O(|y'|^2) + O(y_N) & O(y_N) \\ \hline O(y_N) & 1 + O(|y'|^2) + O(y_N) \end{array}\right).$$

small near the boundary of Γ_r^-

Double approximation procedure

Assume that there exists $p > \frac{N}{2s}$ such that $h \in W^{1,p}(\Omega)$.

- Take a sequence $h_n \in C^\infty\left(\overline{\Gamma_R^-}\right)$ such that $h_n \to \tilde{h}$ in $W^{1,p}(\Gamma_R^-)$.

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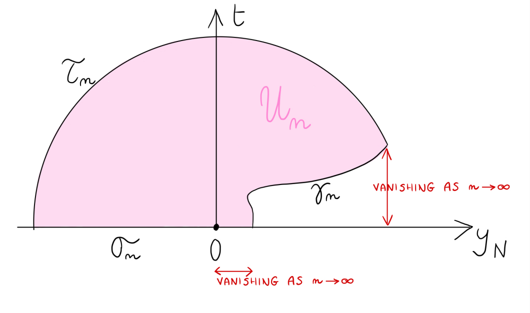
- Construct a sequence of approximating domains $\mathcal{U}_n$ with section like:



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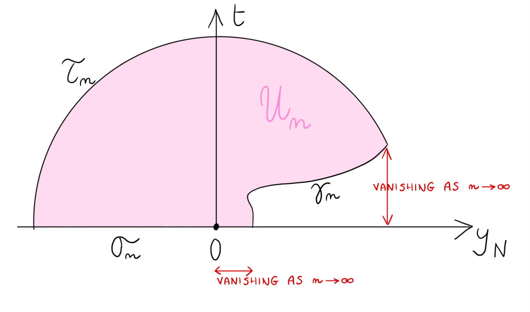
- Construct a sequence of approximating domains $\mathcal{U}_n$ with section like:

  $\forall\ z = (x, t) \in \gamma_n$ and n large

  $$A(y)z \cdot \nu \geq 0 \quad \text{on } \gamma_n.$$



Approximating problems in the domains $\mathcal{U}_n$

$$\begin{cases} -\mathrm{div}\left(t^{1-2s} A \nabla U_n\right) = 0 & \text{in } \mathcal{U}_n, \\ \lim_{t\to 0^+}\left(t^{1-2s} A\nabla U_n \cdot \nu\right) = \kappa_s \eta_n h_n \,\mathrm{Tr}\, U_n & \text{in } \sigma_n, \\ U_n = G_n & \text{in } \tau_n \cup \gamma_n, \end{cases}$$

where

- $G_n \in C_c^\infty(\overline{B_R^+} \setminus \Gamma_R^+)$, $G_n \to W$ strongly in $H^1(B_R^+; t^{1-2s})$ and $G_n = 0$ on γ_n
- η_n are cut-off functions vanishing around $\partial\Gamma_R^-$.

Pohozaev identity for U_n

U_n has enough regularity to integrate a Rellich–Nečas identity $\rightsquigarrow$

$$
\begin{aligned}
&r\int_{\mathcal{U}_n\cap\partial B_r} t^{1-2s} A\nabla U_n\cdot\nabla U_n\,dS - 2r\int_{\mathcal{U}_n\cap\partial B_r} t^{1-2s}\frac{|A\nabla U_n\cdot\nu|^2}{\mu}\,dS\\
&\qquad - 2\kappa_s\int_{\sigma_n\cap B_r}\frac{1}{\mu}\eta_n h_n \mathrm{Tr}\, U_n (D\nabla_y \mathrm{Tr}\, U_n\cdot y)\,dy\\
&= \int_{\mathcal{U}_n\cap B_r} t^{1-2s} A\nabla U_n\cdot\nabla U_n\ \mathrm{div}\beta\,dz - 2\int_{\mathcal{U}_n\cap B_r} t^{1-2s} J_\beta(A\nabla U_n)\cdot\nabla U_n\,dz\\
&\qquad + \int_{\mathcal{U}_n\cap B_r} t^{1-2s}(dA\nabla U_n\nabla U_n)\cdot\beta\,dz + (1-2s)\int_{\mathcal{U}_n\cap B_r} t^{1-2s}\frac{\alpha}{\mu}A\nabla U_n\cdot\nabla U_n\,dz\\
&\qquad\qquad + \int_{\gamma_n\cap B_r}\frac{t^{1-2s}}{\mu}|\partial_\nu U_n|^2(A\nu\cdot\nu)(Az\cdot\nu)\,dS
\end{aligned}
$$

where $\beta(z)=\frac{A(y)z}{\mu(z)}$, $\mu(z)=\frac{A(y)z\cdot z}{|z|^2}$, $\alpha=\det J_F$.

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&\qquad + \int_{\mathcal{U}_n\cap B_r} t^{1-2s}(dA\nabla U_n\nabla U_n)\cdot\beta\,dz + (1-2s)\int_{\mathcal{U}_n\cap B_r} t^{1-2s}\frac{\alpha}{\mu}A\nabla U_n\cdot\nabla U_n\,dz \\
&\qquad\qquad + \underbrace{\int_{\gamma_n\cap B_r}\frac{t^{1-2s}}{\mu}|\partial_\nu U_n|^2(A\nu\cdot\nu)(Az\cdot\nu)\,dS}_{\geq 0}
\end{aligned}
$$

where $\beta(z) = \frac{A(y)z}{\mu(z)}$, $\mu(z) = \frac{A(y)z\cdot z}{|z|^2}$, $\alpha = \det J_F$.

Pohozaev "inequality" for U

$$U_n \to W \text{ strongly in } H^1(B_R^+; t^{1-2s})$$

$$\Downarrow$$

$$\begin{aligned}
&\frac{r}{2}\int_{\partial^+ B_r^+} t^{1-2s} A\nabla W\cdot\nabla W\, dS - r\int_{\partial^+ B_r^+} t^{1-2s}\frac{|A\nabla W\cdot\nu|^2}{\mu}\, dS\\
&\quad+\frac{\kappa_s}{2}\int_{\Gamma_r^-}\left(\nabla\tilde{h}\cdot\beta' + \tilde{h}\,\mathrm{div}\beta'\right)|\mathrm{Tr}W|^2 dy - \frac{\kappa_s r}{2}\int_{S_r^-}\tilde{h}|\mathrm{Tr}W|^2 dS'\\
&\geq \frac{1}{2}\int_{B_r^+} t^{1-2s} A\nabla W\cdot\nabla W\ \mathrm{div}\beta\, dz - \int_{B_r^+} t^{1-2s} J_\beta(A\nabla W)\cdot\nabla W\, dz\\
&\quad+\frac{1}{2}\int_{B_r^+} t^{1-2s}(dA\nabla W\nabla W)\cdot\beta\, dz + \frac{1-2s}{2}\int_{B_r^+} t^{1-2s}\frac{\alpha}{\mu}A\nabla W\cdot\nabla W\, dz
\end{aligned}$$

Frequency function

For small $r > 0$ define

$$D(r) = \frac{1}{r^{N-2s}} \left(\int_{B_r^+} t^{1-2s} A \nabla W \cdot \nabla W \, dz - \kappa_s \int_{\Gamma_r^-} \tilde{h} |\operatorname{Tr} W|^2 \, dy \right)$$

$$H(r) = \frac{1}{r^{N+1-2s}} \int_{S_r^+} t^{1-2s} \mu(z) W^2(z) \, dS$$

where $S_r^+ = \{z = (t, x) \in \partial B_r : t > 0\}$.

Almgren type frequency function

$$\mathcal{N}(r) = \frac{D(r)}{H(r)}$$

well defined for $r > 0$ sufficiently small if $W \not\equiv 0$.

Monotonicity $\rightsquigarrow$ unique continuation

- Our Pohozaev "inequality" $\implies$

$$D'(r) \geq \frac{2}{r^{N-2s}} \int_{S_r^+} t^{1-2s} \frac{|A\nabla W \cdot \nu|^2}{\mu} + O(r^{-1+\delta}) \left[D(r) + \frac{N-2s}{2} H(r) \right] \quad \text{as } r \to 0^+$$

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- Our Pohozaev "inequality" $\Longrightarrow$

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- $\mathcal{N}' \geq$ integrable function: enough to prove the existence of $\gamma = \lim_{r\to 0^+} \mathcal{N}(r)$

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- $\mathcal{N}' \geq$ integrable function: enough to prove the existence of $\gamma = \lim_{r \to 0^+} \mathcal{N}(r)$
- In particular $\mathcal{N}$ is bounded near 0

$$\frac{H'}{H} = \frac{2}{r} \mathcal{N} + O(1) \quad \text{as } r \to 0^+.$$

Integrate between r and $2r$ $\rightsquigarrow$ doubling condition

$$H(2r) \leq CH(r)$$

$\rightsquigarrow$ unique continuation for the extended problem

Monotonicity $\rightsquigarrow$ unique continuation

- Our Pohozaev "inequality" $\implies$

$$D'(r) \geq \frac{2}{r^{N-2s}} \int_{S_r^+} t^{1-2s} \frac{|A\nabla W \cdot \nu|^2}{\mu} + O(r^{-1+\delta}) \left[D(r) + \frac{N-2s}{2} H(r) \right] \quad \text{as } r \to 0^+$$

- $\mathcal{N}' \geq$ integrable function: enough to prove the existence of $\gamma = \lim_{r\to 0^+} \mathcal{N}(r)$
- In particular $\mathcal{N}$ is bounded near 0

$$\frac{H'}{H} = \frac{2}{r}\mathcal{N} + O(1) \quad \text{as } r \to 0^+.$$

Integrate between r and $2r$ $\rightsquigarrow$ doubling condition

$$H(2r) \leq CH(r)$$

$\rightsquigarrow$ unique continuation for the extended problem

but not yet for the original nonlocal problem

Blow-up analysis

$$w^\lambda(z) := \frac{W(\lambda z)}{\sqrt{H(\lambda)}}$$

$\mathcal{N}$ bounded $\Rightarrow$ $\{w^\lambda\}_{\lambda\in(0,R)}$ bounded in $H^1(B_1^+; t^{1-2s})$

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$\mathcal{N}$ bounded $\Rightarrow$ $\{w^\lambda\}_{\lambda\in(0,R)}$ bounded in $H^1(B_1^+;t^{1-2s})$

$$\begin{cases} -\mathrm{div}\left(t^{1-2s}A(\lambda\,\cdot)\nabla w^\lambda\right) = 0 & \text{in } B_1^+ \\ \lim_{t\to0^+}\left(t^{1-2s}A(\lambda\,\cdot)\nabla w^\lambda\cdot\nu\right) = \kappa_s\lambda^{2s}\tilde{h}(\lambda\cdot)\,\mathrm{Tr}\,w^\lambda & \text{on } \Gamma_1^- \\ w^\lambda = 0 & \text{on } \Gamma_1^+ \end{cases}$$

$$\int_{S_1^+}\theta_{N+1}^{1-2s}\mu(\lambda\theta)|w^\lambda(\theta)|^2\,dS = 1$$

$w^\lambda \to w$ in $H^1(B_1^+;t^{1-2s})$, with

$$\begin{cases} -\mathrm{div}\left(t^{1-2s}\nabla w\right) = 0 & \text{in } B_1^+ \\ \lim_{t\to0^+}\left(t^{1-2s}\frac{\partial w}{\partial t}\right) = 0 & \text{on } \Gamma_1^- \\ w = 0 & \text{on } \Gamma_1^+ \end{cases}$$

$$\int_{S_1^+}\theta_{N+1}^{1-2s}w^2(\theta)\,dS = 1$$

Characterization of the limit profiles w

The frequency function associated to w is constantly equal to $\gamma \Rightarrow$

$$w(r\theta) = r^{\gamma}\psi(\theta), \quad r \in (0,1),\ \theta \in \mathbb{S}^N_+$$

where ψ is and eigenfunction of the problem

$$\begin{cases} -\mathrm{div}_{\mathbb{S}^N}\left(\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}^N}\psi\right) = \mu\theta_{N+1}^{1-2s}\psi & \text{in } \mathbb{S}^N_+, \\ \psi = 0 & \text{on } \mathbb{S}^{N-1}\cap\{\theta_N \geq 0\}, \\ \lim_{\theta_{N+1}\to 0^+}\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}^N}\psi\cdot\nu = 0 & \text{on } \mathbb{S}^{N-1}\cap\{\theta_N < 0\}, \end{cases} \qquad (EP_{\mathbb{S}^N_+})$$

on the half-sphere $\mathbb{S}^N_+ = \{(\theta_1,\dots,\theta_N,\theta_{N+1}) \in \mathbb{S}^N : \theta_{N+1} > 0\}$.

Weighted eigenvalue problem on $\mathbb{S}^N_+$ with mixed Dirichlet-Neumann b. c.

$$\begin{cases} -\mathrm{div}_{\mathbb{S}^N}\left(\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}^N}\psi\right) = \mu\theta_{N+1}^{1-2s}\psi & \text{in } \mathbb{S}^N_+, \\ \psi = 0 & \text{on } \mathbb{S}^{N-1}\cap\{\theta_N \geq 0\}, \\ \lim_{\theta_{N+1}\to 0^+}\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}^N}\psi\cdot\nu = 0 & \text{on } \mathbb{S}^{N-1}\cap\{\theta_N < 0\}, \end{cases} \qquad (EP_{\mathbb{S}^N_+})$$

Classical spectral theory $\rightsquigarrow$ $\exists$ a diverging sequence $\{\mu_k\}_{k\in\mathbb{N}}$ of real eigenvalues with finite multiplicity M_k

$$\mu_k = (k+s)(k+N-s), \quad k\in\mathbb{N}.$$

Blow-up analysis $\to$ quantization of possible vanishing orders $\to$ SUCP

Come back to $U = W \circ F^{-1}$:

Theorem [De Luca-F.-Vita (2021)]

Let $U \not\equiv 0$ be such that $U = \mathcal{H}(u)$ with u satisfying (E_s). Then there exists $k_0 \in \mathbb{N}$ and an eigenfunction Y of problem $(EP_{\mathbb{S}^N_+})$ associated to the eigenvalue $\mu_{k_0} = (k_0 + s)(k_0 + N - s)$ such that

$$\frac{U(\lambda z)}{\lambda^{k_0+s}} \to |z|^{k_0+s} Y\left(\frac{z}{|z|}\right) \quad \text{in } H^1(B_1^+; t^{1-2s}) \quad \text{as } \lambda \to 0^+.$$

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$$\Downarrow$$

SUCP for U

If $U = \mathcal{H}(u)$ with u satisfying (E_s) and $U(z) = O(|z|^k)$ as $z \to 0$, for any $k \in \mathbb{N}$, then $U \equiv 0$ in $\mathbb{R}^{N+1}_+$.

Asymptotics and SUCP for the fractional problem

Theorem [De Luca-F.-Vita (2021)]

Let Ω be a bounded domain in $\mathbb{R}^N$, $x_0 \in \partial\Omega$ s.t. $\partial\Omega$ is $C^{1,1}$ in a neighbourhood of x_0.
Let $h \in W^{1,p}(\Omega)$ for some $p > \frac{N}{2s}$ and let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, $u \not\equiv 0$, be a weak solution to (E_s).

Then there exists $k_0 \in \mathbb{N}$ and an eigenfunction Y of problem $(EP_{\mathbb{S}^N_+})$ associated to the eigenvalue $\mu_{k_0} = (k_0 + s)(k_0 + N - s)$ such that

$$\frac{u(x_0 + \lambda x)}{\lambda^{k_0+s}} \to |x|^{k_0+s} Y\left(\frac{x}{|x|}, 0\right) \quad \text{in } H^s(B_1') \text{ as } \lambda \to 0^+.$$

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$$\Downarrow$$

SUCP for u

If $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ is a weak solution to (E_s) such that $u(x) = O(|x - x_0|^k)$ as $x \to x_0$ for any $k \in \mathbb{N}$, then $u \equiv 0$ in $\mathbb{R}^N$.