Fractional semidiscrete evolution equations in Lebesgue sequence spaces Advances in Difference Equations, (2021)

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Fundamental solutions for semidiscrete evolution equations via Banach algebras

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Abstract

We give representations for solutions of time-fractional differential equations that involve operators on Lebesgue spaces of sequences defined by discrete convolutions involving kernels through the discrete Fourier transform. We consider finite difference operators of first and second orders, which are generators of uniformly continuous semigroups and cosine functions. We present the linear and algebraic structures (in particular, factorization properties) and their norms and spectra in the Lebesgue space of summable sequences. We identify fractional powers of these generators and apply to them the subordination principle. We also give some applications and consequences of our results.

MSC: 35R11; 35A08; 39A12

Keywords: Caputo fractional derivative; Discrete fractional Laplacian; Discrete fractional operators; Fundamental solutions; Wright and Mittag-Leffler functions

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Jorge González-Camus (USACH)   Carlos Lizama (USACH)

1. Introduction

We study the fractional differential equation

$$\begin{cases} \mathbb{D}_t^{\beta} u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0. \\ u(n,0) = \varphi(n), \ \ u_t(n,0) = \phi(n) & n \in \mathbb{Z}, \end{cases} \tag{1}$$

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$Bf(n) = (b * f)(n)$, with $b \in l^1(\mathbb{Z})$, $f \in l^p(\mathbb{Z})$, $p \in [1,\infty]$ and $\beta \in (0,2]$. We recall that $\mathbb{D}_t^{\beta}$ denotes the Caputo fractional derivative.

For a regular function v, $\mathbb{D}_t^\beta$ is the Caputo derivative of order β,

$$\mathbb{D}_t^\beta v(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} v'(s) ds = (g_{1-\beta} * v')(t), \qquad t > 0,$$

for $0 < \beta < 1$ and

$$\mathbb{D}_t^\beta v(t) = \frac{1}{\Gamma(2-\beta)} \int_0^t (t-s)^{1-\beta} v''(s) ds = (g_{2-\beta} * v'')(t), \qquad t > 0,$$

for $1 < \beta < 2$.

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for $1 < \beta < 2$. For $\beta = 1$ and $\beta = 2$, note that

$$\lim_{\beta \to 1^-} \mathbb{D}_t^\beta v(t) = v'(t), \qquad \lim_{\beta \to 2^-} \mathbb{D}_t^\beta v(t) = v''(t), \qquad t > 0,$$

however,

$$\lim_{\beta \to 0^+} \mathbb{D}_t^\beta v(t) = v(t) - v(0), \qquad \lim_{\beta \to 1^+} \mathbb{D}_t^\beta v(t) = v'(t) - v'(0), \qquad t > 0,$$

see, for example [Baz, GM].

For $\beta = 1$, the semi discrete Cauchy problem given in the introduction

$$\begin{cases} \partial_t u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}, \end{cases}$$

and their fundamental solution that is obviously given by Duhamel's formula

$$u(n,t) = e^{Bt}\varphi(n) + \int_0^t e^{B(t-s)} g(n,s)ds \quad n \in \mathbb{Z}, \quad t \geq 0.$$


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Jean Marie Duhamel (1797 - 1872)

His studies were affected by the troubles of the Napoleonic era (Gaspar Monge). Duhamel worked on partial differential equations and applied his methods to the theory of heat, to rational mechanics, and to acoustics. He was an experimenter and published several memories. *Sur la méthode générale relative au mouvement de la chaleur dans les corps solides plongés dans des milieux dont la température varie avec le temps* (1833).

UNIVERSITE PARIS VI

SEMINAIRE DE

THEORIE DU POTENTIEL

FORMULE DE DUHAMEL ET PROBLEME DE LA CHALEUR

par Luc PAQUET ★

0. INTRODUCTION

Dans un mémoire en 1833, J.M.C. Duhamel établit par des considérations heuristiques le théorème suivant [Duh],[C-J] : "notons $F(x,y,z,\lambda,t)$ la température au point (x,y,z) à l'instant t dans un solide dont la température initiale est nulle et sa température au bord $\phi(x,y,z,\lambda)$. Alors la solution du problème dans lequel la température initiale est zéro et la température au bord $\phi(x,y,z,t)$ est donnée par :

$$v(x,y,z,t) = \int_0^t \frac{\partial}{\partial t} F(x,y,z,\lambda,t-\lambda)\,d\lambda \text{ ".} \qquad (1)$$

Analogously, in the case of the second order semi discrete Cauchy problem:

$$\begin{cases} \partial_{tt} u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z}, \quad t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}, \\ u_t(n,0) = \phi(n), & n \in \mathbb{Z}, \end{cases}$$

we have that the fundamental solution is given by D'Alembert formula

$$u(n,t) = \mathrm{Cos}(t,B)\varphi(n) + \mathrm{Sin}(t,B)\phi(n) + \int_0^t \mathrm{Sin}(t-s,B) f(s) ds,$$

where $\mathrm{Cos}(t,B)$ and $\mathrm{Sin}(t,B)$ are generated by B.

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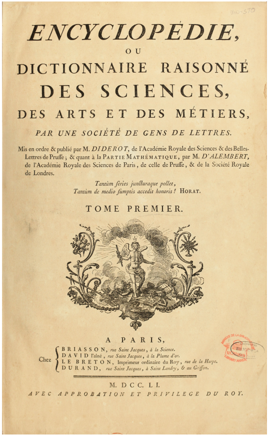


ENCYCLOPÉDIE,
OU
DICTIONNAIRE RAISONNÉ
DES SCIENCES,
DES ARTS ET DES MÉTIERS,
PAR UNE SOCIÉTÉ DE GENS DE LETTRES.

Mis en ordre & publié par M. DIDEROT, de l'Académie Royale des Sciences & des Belles-Lettres de Pruſſe; & quant à la PARTIE MATHÉMATIQUE, par M. D'ALEMBERT, de l'Académie Royale des Sciences de Paris, de celle de Pruſſe, & de la Société Royale de Londres.

Tantùm ſeries juncturaque pollet,
Tantùm de medio ſumptis accedit honoris! HORAT.

TOME PREMIER.

A PARIS,
Chez BRIASSON, rue Saint Jacques, à la Science.
DAVID l'aîné, rue Saint Jacques, à la Plume d'or.
LE BRETON, Imprimeur ordinaire du Roy, rue de la Harpe.
DURAND, rue Saint Jacques, à Saint Landry, & au Griffon.

M. DCC. LI.
AVEC APPROBATION ET PRIVILEGE DU ROY.

Jean Le Rond d'Alembert (1717 - 1783)

He was left the newly born child on the steps of the church St Jean Le Rond. D'Alembert worked on the Encyclopédie for many years, (28 volumes). He was also a brilliant mathematician. He felt that Euler was stealing his ideas and not giving him due credit. He stopped publishing his articles, collecting them in *Opuscules mathématiques* (8 volumes).

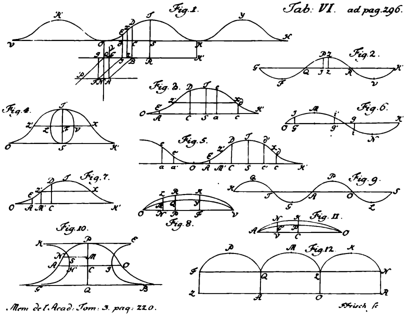
RECHERCHES
SUR LA COURBE QUE FORME UNE CORDE
TENDUË MISE EN VIBRATION,

PAR MR. D'ALEMBERT.

I.

Je me propose de faire voir dans ce Memoire, qu'il y a une infinité d'autres courbes que *la Compagne de la Cycloide allongée*, qui satisfont au Probleme dont il s'agit. Je supposeray toujours 1mo, que les excursions ou vibrations de la corde sont fort petites, ensorte que
Fig. 1. les arcs A M de la courbe qu'elle forme, puissent toujours être supposés sensiblement égaux aux abscisses correspondantes A P. 2°. que la corde est uniformement epaisse dans toute sa longueur: 3°. que la force F de la tension est au poids de la corde, en raison constante, c. a. d. comme m à 1; d'où il s'ensuit que si on nomme p la gravité, & l la longueur de la corde, on pourra supposer $F = p\,m\,l$; 4°. que si on nomme A P ou A M, s; P M, y; & qu'on fasse ds constante, la force acceleratrice du point M suivant M P, est $-\frac{F\,ddy}{ds^2}$, si la courbe est concave vers A C, ou $\frac{F\,ddy}{ds^2}$ si elle est convexe. *Voyez Taylor Meth. Incr.*

II. Cela



Aims of the talk

Aims of the talk

The main aim of this talk is to study the fractional differential equations in $\ell^p(\mathbb{Z})$ for $1 \leq p \leq \infty$. To do this.

(i) We apply Güelfand theory to describe convolution operators.

(ii) We calculate the kernel of the convolution fractional powers.

(iii) We solve some fractional evolution equation in $\ell^p(\mathbb{Z})$.

(iv) We obtain explict solutions for fractional evolution equation for some fractional powers of finite difference operators.

(v) Finally we give some application to concrete equations and special functions.

2. A Banach algebra framework

For $1 \leq p \leq \infty$, the Banach space $(\ell^p(\mathbb{Z}), \| \quad \|_p)$ are formed by $f = (f(n))_{n\in\mathbb{Z}} \subset \mathbb{C}$ such that

$$
\begin{aligned}
\|f\|_p : &= \left(\sum_{n=-\infty}^{\infty} |f(n)|^p \right)^{\frac{1}{p}} < \infty, \qquad 1 \leq p < \infty; \\
\|f\|_\infty : &= \sup_{n\in\mathbb{Z}} |f(n)| < \infty.
\end{aligned}
$$

$\ell^1(\mathbb{Z}) \hookrightarrow \ell^p(\mathbb{Z}) \hookrightarrow \ell^\infty(\mathbb{Z})$, $(\ell^p(\mathbb{Z}))' = \ell^{p'}(\mathbb{Z})$ with $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 < p < \infty$ and $p = 1$ and $p' = \infty$.

2. A Banach algebra framework

In the case that $f \in \ell^1(\mathbb{Z})$ and $g \in \ell^p(\mathbb{Z})$, then $f * g \in \ell^p(\mathbb{Z})$ where

$$(f * g)(n) := \sum_{j=-\infty}^{\infty} f(n-j)g(j), \qquad n \in \mathbb{Z},$$

and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$ for $1 \leq p \leq \infty$. Note that $(\ell^1(\mathbb{Z}), *)$ is a commutative Banach algebra with unit (we write $\delta_0 = \chi_{\{0\}}$).

2. A Banach algebra framework

We recall that the spectrum of f, denoted as $\sigma_{\ell^1(\mathbb{Z})}(f)$, is defined by $\sigma_{\ell^1(\mathbb{Z})}(f) := \mathbb{C}\backslash\rho_{\ell^1(\mathbb{Z})}(f)$, where

$$\rho_{\ell^1(\mathbb{Z})}(f) := \{\lambda \in \mathbb{C} \ : \ (\lambda\delta_0 - f)^{-1} \in \ell^1(\mathbb{Z})\}.$$

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We apply Güelfand theory to get

$$\sigma_{\ell^1(\mathbb{Z})}(f) = \mathcal{F}(f)(\mathbb{T}), \qquad f \in \ell^1(\mathbb{Z}),$$

where $\mathcal{F} : \ell^1(\mathbb{Z}) \to \mathcal{C}(\mathbb{T})$ is the discrete Fourier transform,

$$\mathcal{F}(f)(\theta) := \sum_{n\in\mathbb{Z}} f(n)e^{in\theta}, \quad \theta \in \mathbb{T}.$$

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$$\mathcal{F}(f)(\theta) := \sum_{n\in\mathbb{Z}} f(n)e^{in\theta}, \quad \theta \in \mathbb{T}.$$

The inverse discrete Fourier transform is given by

$$\mathcal{F}^{-1}(F)(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{i\theta})e^{-in\theta}\, d\theta = \frac{1}{2\pi i}\int_{|z|=1} F(z)\frac{dz}{z^{n+1}}, \quad n \in \mathbb{Z},$$

for $F \in \mathcal{A}(\mathbb{T})$ (and for other functions in larger sets).

2. A Banach algebra framework

Definition

For $\alpha, \beta > 0$, the vector-valued Mittag-Leffler function, $E_{\alpha,\beta} : \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})$,

$$E_{\alpha,\beta}(a) := \sum_{j=0}^{\infty} \frac{a^j}{\Gamma(\alpha j + \beta)}, \qquad a \in \ell^1(\mathbb{Z}).$$

Note that

$$E_{1,1}(a) = \sum_{j=0}^{\infty} \frac{a^j}{j!} = e^a; \qquad E_{2,1}(a) = \sum_{j=0}^{\infty} \frac{a^j}{(2j)!}.$$

The element a is the generator of the entire group $(e^{za})_{z\in\mathbb{C}}$; a cosine function, $\mathrm{Cos}(z, a) := E_{2,1}(z^2 a)$, and a sine function, $\mathrm{Sin}(z, a) := zE_{2,2}(z^2 a)$. We have

$$\mathrm{Sin}(z, a) = \int_{[0,z]} \mathrm{Cos}(s, a)ds, \qquad z \in \mathbb{C}, \quad a \in \ell^1(\mathbb{Z}).$$

2. A Banach algebra framework

For $\nu \in \mathbb{R}$, let J_ν denote the *Bessel function* defined by

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\nu+1)n!} \left(\frac{x}{2}\right)^{2n+\nu}, \quad x \geq 0. \tag{2}$$

the *Modified Bessel functions of the first kind*, defined by

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\nu+1)n!} \left(\frac{x}{2}\right)^{2n+\nu}. \tag{3}$$

Note that $|J_\nu(x)| \leq I_\nu(x), \nu \in \mathbb{R}_+$ and $I_n(x) \geq 0,\ n \in \mathbb{Z},\ x \geq 0$,

1. $\sum_{n\in\mathbb{Z}} J_n(x) z^n = e^{\frac{x}{2}(z-\frac{1}{z})}, \quad z \in \mathbb{C} \setminus \{0\}, \quad x \in \mathbb{C}.$
2. $J_n(x+y) = \sum_{k\in\mathbb{Z}} J_{n-k}(x) J_k(y).$
3. $\sum_{n\in\mathbb{Z}} I_n(x) z^n = e^{\frac{x}{2}(z+\frac{1}{z})}, \quad z \in \mathbb{C} \setminus \{0\}, \quad x \in \mathbb{C}.$
4. $I_n(x+y) = \sum_{k\in\mathbb{Z}} I_{n-k}(x) I_k(y).$

2. A Banach algebra framework

The Laplace transform of a entire group or a cosine function is connected with the resolvent of its generator:

$$\begin{aligned}(\lambda - a)^{-1} &= \int_0^\infty e^{-\lambda s} e^{as} ds, \qquad \lambda > \|a\|_1, \\ \lambda(\lambda^2 - a)^{-1} &= \int_0^\infty e^{-\lambda s} \mathrm{Cos}(s, a) ds, \qquad \lambda > \sqrt{\|a\|_1}.\end{aligned}$$

Example

For $\alpha, \beta > 0$, we have that

$$E_{\alpha,\beta}(z\delta_0) = E_{\alpha,\beta}(z)\delta_0; \qquad E_{\alpha,\beta}(z\delta_1) = \sum_{j=0}^{\infty} \frac{z^j \delta_j}{\Gamma(\alpha j + \beta)}.$$

In particular, $e^{z\delta_1} = \sum_{j=0}^{\infty} \frac{z^j \delta_j}{j!}$ and $\mathrm{Cos}(z, \delta_1) = \sum_{j=0}^{\infty} \frac{z^{2j} \delta_j}{(2j)!}$ are generated by δ_1.

2. A Banach algebra framework

Proposition

For $\alpha, \beta > 0$ and $a \in \ell^1(\mathbb{Z})$, we have that

(i) $\|E_{\alpha,\beta}(a)\|_1 \leq E_{\alpha,\beta}(\|a\|_1)$.

(ii) $\mathcal{F}(E_{\alpha,\beta}(a)) = E_{\alpha,\beta}(\mathcal{F}(a))$; *in particular* $\mathcal{F}(e^{az}) = e^{z\mathcal{F}(a)}$ *and* $\mathcal{F}(Cos(z,a)) = Cos(\mathcal{F}(z), a)$ *for* $z \in \mathbb{C}$.

(iii) $\sigma_{\ell^1(\mathbb{Z})}(E_{\alpha,\beta}(a)) = E_{\alpha,\beta}(\sigma_{\ell^1(\mathbb{Z})}(a))$.

(iv) *The following Laplace transform formula holds*

$$\int_0^\infty e^{-\lambda t} t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(t^\alpha a)dt = k!\lambda^{\alpha-\beta}\left((\lambda^\alpha - a)^{-1}\right)^{(k+1)},$$

for $\Re(\lambda) > \|a\|_1^{1/\alpha}$, *and* $k \in \mathbb{N} \cup \{0\}$.

2. A Banach algebra framework

Given $a = (a(n))_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, define $A \in \mathcal{B}(\ell^p(\mathbb{Z}))$ by convolution,

$$A(b)(n) := (a * b)(n), \qquad n \in \mathbb{Z}, \qquad b \in \ell^p(\mathbb{Z}),$$

for all $1 \leq p \leq \infty$, $\|A\| = \|a\|_1$ and

$$\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(A) = \sigma_{\ell^1(\mathbb{Z})}(a) = \mathcal{F}(a)(\mathbb{T}) \tag{4}$$

for all $1 \leq p \leq \infty$, (Wiener's Lemma).

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for all $1 \leq p \leq \infty$, (Wiener's Lemma).

It is also straightforward to check that the adjoint operator of A is again a convolution operator given by $A'(g)(n) := (\tilde{a} * g)(n)$ where

$$\tilde{a}(n) = a(-n), \qquad n \in \mathbb{Z}.$$

3. Some finite difference operators in $\ell^1(\mathbb{Z})$

Finite difference operators $A \in \mathcal{B}(\ell^p(\mathbb{Z}))$ given by

$$Af(n) := \sum_{j=-m}^{m} a(j) f(n-j), \quad a_j \in \mathbb{C},$$

for some $m \in \mathbb{N}$, i.e. $a = (a(n)_{n\in\mathbb{Z}}) \in c_c(\mathbb{Z})$ are convolution operator and the discrete Fourier Transform of a is a trigonometric polynomial

$$\mathcal{F}(a)(\theta) = \sum_{j=-m}^{m} a(j) e^{ij\theta}.$$

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$$\mathcal{F}(a)(\theta) = \sum_{j=-m}^{m} a(j)e^{ij\theta}.$$

It is interesting to observe that if $\sum_{j=-m}^{m} a(j) = 0$ then $0 \in \sigma_{\ell^1(\mathbb{Z})}(a)$.

3. Some finite difference operators in $\ell^1(\mathbb{Z})$

Definition

[B] For $f \in \ell^p(\mathbb{Z})$, with $1 \leq p \leq \infty$, we define operators

1. $-\Delta f(n) := f(n) - f(n+1) = ((\delta_0 - \delta_{-1}) * f)(n)$;
2. $\nabla f(n) := f(n) - f(n-1) = ((\delta_0 - \delta_1) * f)(n)$;
3. $\Delta_d f(n) := f(n+1) - 2f(n) + f(n-1) = ((\delta_{-1} - 2\delta_0 + \delta_1) * f)(n)$;
4. $\Delta_{dd} f(n) := f(n+2) - 2f(n) + f(n-2) = ((\delta_{-2} - 2\delta_0 + \delta_2) * f)(n)$;

for $n \in \mathbb{Z}$.

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for $n \in \mathbb{Z}$.

Operators $-\Delta$ and ∇ are related to Euler scheme of approximation, and the operator Δ_d corresponds to the second-order central difference approximation for the second order derivative. The operator Δ_{dd} appears in Bateman's paper in connection with the equations of Born and Karman on crystal lattices in vibration.

3. Some finite difference operators in $\ell^1(\mathbb{Z})$

Harry Bateman (1882 – 1946)

3. Some finite difference operators in $\ell^1(\mathbb{Z})$

Theorem

The operator $-\Delta f = a * f$ *where* $a := \delta_0 - \delta_{-1}$ *verifies*

1. *The norm is given by* $\|\Delta\| = 2$;
2. *The Fourier transform is* $\mathcal{F}(a)(z) = 1 - z, \ |z| = 1$;
3. *For all* $1 \leq p \leq \infty$, $\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(-\Delta) = \{z \in \mathbb{T} \ : |z-1| = 1\}$;
4. *For* $|\lambda + 1| > 1$, $(\lambda\delta_0 + a)^{-1} = \sum_{j\geq 0} \frac{\delta_{-j}}{(1+\lambda)^{j+1}}$.
5. *The associated group is* $e^{-za}(n) = e^{-z} \frac{z^{-n}}{(-n)!} \chi_{-\mathbb{N}_0}(n)$, *for* $z \in \mathbb{C}$, $n \in \mathbb{Z}$ *and its generator is* $-a$.
6. *The norm of the group is given by* $\|e^{-ta}\|_1 = 1, \quad t > 0$;
7. *The associated cosine function is*

$$Cos(z, -a)(n) = \frac{\sqrt{\pi}}{(-n)!} \left(\frac{z}{2}\right)^{-n+\frac{1}{2}} J_{-n-\frac{1}{2}}(z) \chi_{-\mathbb{N}_0}(n), z \in \mathbb{C}, \ n \in \mathbb{Z}.$$

3. Some finite difference operators in $\ell^1(\mathbb{Z})$

Theorem

The operator $\nabla f = a * f$ *where* $a := \delta_0 - \delta_1$ *verifies*

1. $\|\nabla\| = 2$;
2. $\mathcal{F}(a)(z) = 1 - \dfrac{1}{z}$;
3. *For* $1 \leq p \leq \infty$, $\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(\nabla) = \{z \in \mathbb{T} \ : |z-1| = 1\}$;
4. *For* $|\lambda + 1| > 1$,

$$(\lambda\delta_0 + a)^{-1} = \sum_{j\geq 0} \frac{\delta_j}{(1+\lambda)^{j+1}}.$$

5. $e^{-za}(n) = e^{-z}\frac{z^n}{n!}\chi_{\mathbb{N}_0}(n), \quad z \in \mathbb{C}, \quad n \in \mathbb{Z}$;
6. $\|e^{-ta}\|_1 = 1, \quad t > 0$;
7. $Cos(z, -a) = \frac{\sqrt{\pi}}{n!}\left(\frac{z}{2}\right)^{n+\frac{1}{2}} J_{n-\frac{1}{2}}(z)\chi_{\mathbb{N}_0}(n), \ z \in \mathbb{C}, \ n \in \mathbb{Z}.$

3. Some finite difference operators in $\ell^1(\mathbb{Z})$

Theorem

The operator $\Delta_d f = a * f$ *where* $a := \delta_{-1} - 2\delta_0 + \delta_1$ *verifies*

1. $\|\Delta_d\| = 4$;
2. $\mathcal{F}(a)(z) = z + \frac{1}{z} - 2$;
3. *For all* $1 \leq p \leq \infty$ *we have* $\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(\Delta_d) = [-4, 0]$;
4. *The group* $e^{za}(n) = e^{-2z} I_n(2z), \quad z \in \mathbb{C}, \quad n \in \mathbb{Z}$ *and its generator is a.*
5. $\|e^{ta}\|_1 = 1, \quad t > 0$;
6. *For* $\lambda \in \mathbb{C} \setminus [-4, 0]$,

$$(\lambda - a)^{-1}(n) = 2^{-n} \frac{((\lambda + 2) - \sqrt{\lambda^2 + 4\lambda})^n}{\sqrt{\lambda^2 + 4\lambda}}, \qquad n \in \mathbb{Z};$$

7. $Cos(z, a) = J_{2n}(2z), \; z \in \mathbb{C}, \; n \in \mathbb{Z}.$

3. Some finite difference operators in $\ell^1(\mathbb{Z})$

Theorem

The operator $\Delta_{dd} f = a * f$ *where* $a := \delta_{-2} - 2\delta_0 + \delta_2$ *verifies*

1. $\|\Delta_{dd}\| = 4$;
2. $\mathcal{F}(a)(z) = (z - \frac{1}{z})^2$;
3. *For all* $1 \leq p \leq \infty$ *we have* $\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(\Delta_{dd}) = [-4, 0]$;
4. $e^{za}(n) = e^{-2z} I_{\frac{n}{2}}(2z)\chi_{2\mathbb{Z}}(n), \quad z \in \mathbb{C}, \quad n \in \mathbb{Z}$;
5. $\|e^{-ta}\|_1 = 1, \quad t > 0$;
6. *For* $\lambda \in \mathbb{C} \setminus [-4, 0]$,
$$(\lambda - a)^{-1}(n) = 2^{-\frac{n}{2}} \frac{((\lambda + 2) - \sqrt{\lambda^2 + 4\lambda})^{\frac{n}{2}}}{\sqrt{\lambda^2 + 4\lambda}} \chi_{2\mathbb{Z}}(n), \qquad n \in \mathbb{Z};$$
7. $Cos(z, -a)(n) = J_n(2z)\chi_{2\mathbb{Z}}(n), \; z \in \mathbb{C}, \; n \in \mathbb{Z}$.

3. Some finite difference operators in $\ell^1(\mathbb{Z})$

Some simple computations show linear, algebraic and dual relations between the operators defined previously, which are presented in the following result.

Proposition

Let $-\Delta, \nabla, \Delta_d$ and Δ_{dd} be the discrete operators.

(i) *The following equalities hold:*

$$-\Delta_d = (\nabla - \Delta) = -\Delta\nabla.$$

(ii) *For $1 \leq p < \infty$, the following identities hold on $\ell^p(\mathbb{Z})$:*

$$(-\Delta)' = \nabla; \qquad (\nabla)' = -\Delta;$$
$$(\Delta_d)' = \Delta_d; \qquad (\Delta_{dd})' = \Delta_{dd}.$$

3. Some finite difference operators in $\ell^1(\mathbb{Z})$

$$-\Delta_d = (\nabla - \Delta) = -\Delta\nabla.$$

3. Some finite difference operators in $\ell^1(\mathbb{Z})$

$$-\Delta_d = (\nabla - \Delta) = -\Delta\nabla.$$

$$g_{z,-}(n) := \frac{z^n}{n!}\chi_{\mathbb{N}_0}(n), \qquad g_{z,+}(n) := \frac{z^{-n}}{(-n)!}\chi_{-\mathbb{N}_0}(n).$$

Theorem

The Bessel function I_n admits a factorization via convolution given by

$$I_n(2z) = (g_{z,+} * g_{z,-})(n), \qquad n \in \mathbb{Z}, \;\; z \in \mathbb{C}.$$

3. Some finite difference operators in $\ell^1(\mathbb{Z})$

$a * f$	a	$\mathcal{F}(\cdot)(z)$	$\sigma_{\ell^1(\mathbb{Z})}(a)$
$-\Delta$	$\delta_0 - \delta_{-1}$	$1 - z$	$\{z \in \mathbb{T} \ : \lvert z - 1\rvert = 1\}$
∇	$\delta_0 - \delta_1$	$1 - \frac{1}{z}$	$\{z \in \mathbb{T} \ : \lvert z - 1\rvert = 1\}$
Δ_d	$\delta_{-1} - 2\delta_0 + \delta_1$	$z + \frac{1}{z} - 2$	$[-4, 0]$
Δ_{dd}	$\delta_{-2} - 2\delta_0 + \delta_2$	$z^2 - 2 + \frac{1}{z^2}$	$[-4, 0]$

$a * f$	Generated semigroup	Generated cosine
$-\Delta$	$e^{-z}\frac{z^{-n}}{(-n)!}\chi_{-\mathbb{N}_0}(n)$	$\frac{\sqrt{\pi}}{(-n)!}\left(\frac{z}{2}\right)^{-n+\frac{1}{2}} J_{-n-\frac{1}{2}}(z)\chi_{-\mathbb{N}_0}(n)$
∇	$e^{-z}\frac{z^n}{n!}\chi_{\mathbb{N}_0}(n)$	$\frac{\sqrt{\pi}}{n!}\left(\frac{z}{2}\right)^{n+\frac{1}{2}} J_{n-\frac{1}{2}}(z)\chi_{\mathbb{N}_0}(n)$
Δ_d	$e^{-2z} I_n(2z)$	$J_{2n}(2z)$
Δ_{dd}	$e^{-2z} I_{\frac{n}{2}}(2z)\chi_{2\mathbb{Z}}(n)$	$J_n(2z)\chi_{2\mathbb{Z}}(n)$

4. Fractional powers of generators of semigrupos in $\ell^1(\mathbb{Z})$

To define fractional powers in a Banach algebra (and in operator theory) is, in general, a difficult task. Not every element in $\ell^1(\mathbb{Z})$ has fractional powers. For example δ_1 does not have square root in $\ell^1(\mathbb{Z})$.

4. Fractional powers of generators of semigrupos in $\ell^1(\mathbb{Z})$

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When $\sigma_{\ell^1(\mathbb{Z})}(a) \subset \mathbb{C}^+$ and $\alpha \in \mathbb{R}$, we may consider the function $F_\alpha(z) = z^\alpha$ which is holomorphic in a neighbour of $\sigma_{\ell^1(\mathbb{Z})}(a)$. By the analytic functional calculus, the element

$$F_\alpha(a) = \frac{1}{2\pi i} \int_\gamma \frac{F_\alpha(z)}{z - a} dz,$$

(where γ is a spectral contour lying in an open set $\mathcal{O}$ containing the spectrum of a) exists in the Banach algebra $\ell^1(\mathbb{Z})$ and $\mathcal{F}(F_\alpha(a)) = (\mathcal{F}(a))^\alpha$ ([La]). Then $F_\alpha(a)$ is a fractional power of a of order α, and we write $F_\alpha(a) = a^\alpha$.

4. Fractional powers of generators of semigrupos in $\ell^1(\mathbb{Z})$

We note that there exists a classical way to define fractional powers of generators of uniformly bounded semigroups in Banach spaces, see for example [Y] and, in particular in $\ell^1(\mathbb{Z})$.

Definition

Let $0 < \alpha < 1$, and $a \in \ell^1(\mathbb{Z})$, such that $(e^{ta})_{t\geq 0}$ is a uniformly bounded semigroup, i.e., $\sup_{s>0} \|e^{as}\|_1 < \infty$. Then we write $(-a)^\alpha$ by the fractional power of a given by the following integral representation,

$$(-a)^\alpha := \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{e^{sa} - \delta_0}{s^{1+\alpha}} ds.$$

4. Fractional powers of generators of semigrupos in $\ell^1(\mathbb{Z})$

As an immediate consequence of this definition, we have that, for $0 < \alpha < 1$,

$$\mathcal{F}((-a)^\alpha) = (-\mathcal{F}(a))^\alpha, \qquad \sigma((-a)^\alpha) = (\sigma(-a))^\alpha,$$

4. Fractional powers of generators of semigrupos in $\ell^1(\mathbb{Z})$

As an immediate consequence of this definition, we have that, for $0 < \alpha < 1$,

$$\mathcal{F}((-a)^\alpha) = (-\mathcal{F}(a))^\alpha, \qquad \sigma((-a)^\alpha) = (\sigma(-a))^\alpha,$$

It is well known that the uniformly bounded semigroup $(e^{-t(-a)^\alpha})_{t\geq 0}$ is subordinated to $(e^{ta})_{t\geq 0}$ (principle of Lévy subordination) by the formula

$$e^{-t(-a)^\alpha} = \int_0^\infty f_{t,\alpha}(s) e^{as} ds = \sum_{j=0}^\infty \frac{(-t)^j}{j!}(-a)^{j\alpha}, \qquad t \geq 0,$$

see, for example [Y]. Note that

$$\mathcal{F}(e^{-t(-a)^\alpha}) = e^{-t(-\mathcal{F}(a))^\alpha}.$$

4. Fractional powers of generators of semigrupos in $\ell^1(\mathbb{Z})$

$$k^\alpha(j) := \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)j!} \text{ for } j \in \mathbb{N}_0. \quad \sum_{j=0}^{\infty} k^\alpha(j) z^j = \frac{1}{(1-z)^\alpha} \text{ ([Z]).}$$

Fractional power	$\mathcal{F}(\cdot)(z)$	Explicit expression
$(\delta_0 - \delta_{-1})^\alpha$	$(1-\frac{1}{z})^\alpha$	$k^{-\alpha}(n)\chi_{-\mathbb{N}_0}$
$(\delta_0 - \delta_1)^\alpha$	$(1-z)^\alpha$	$k^{-\alpha}(n)\chi_{\mathbb{N}_0}$
$(-(\delta_{-1} - 2\delta_0 + \delta_1))^\alpha$	$(4\sin^2(\frac{\theta}{2}))^\alpha$	$\frac{(-1)^j\Gamma(2\alpha+1)}{\Gamma(1+\alpha+n)\Gamma(1+\alpha-n)}$
$(-(\delta_{-2} - 2\delta_0 + \delta_2))^\alpha$	$(4\sin^2(\theta))^\alpha$	$\frac{\Gamma(2\alpha+1)\cos(\frac{n}{2}\pi)}{\Gamma(1+\alpha+\frac{n}{2})\Gamma(1+\alpha-\frac{n}{2})}$

4. Fractional powers of generators of semigrupos in $\ell^1(\mathbb{Z})$

Now we apply the Lévy subordination principle to $a = \delta_{-1} - \delta_0$ and $a = \delta_1 - \delta_0$.

Corollary

Let $0 < \alpha < 1$ and $f_{s,\alpha}$ is the Lévy stable process. Then

$$\sum_{j=1}^{\infty} k^{-\alpha j}(n)\frac{(-t)^j}{j!} = \int_0^{\infty} f_{t,\alpha}(s)\frac{e^{-s}s^n}{n!}ds, \quad t > 0,\ n \geq 1.$$

In particular, when $\alpha = \frac{1}{2}$, we obtain

$$\sum_{j=1}^{\infty} k^{\frac{-j}{2}}(n)\frac{(-t)^j}{j!} = \int_0^{\infty} \frac{t}{\sqrt{4\pi s^3}} e^{\frac{-t^2}{4s}} e^{-s} s^n ds, \qquad n \geq 1.$$

4. Fractional powers of generators of semigrupos in $\ell^1(\mathbb{Z})$

$$K_d^{\alpha}(n) := \frac{(-1)^n \Gamma(2\alpha+1)}{\Gamma(1+\alpha+n)\Gamma(1+\alpha-n)}.$$

We also apply the Lévy subordination principle to the semigroup generated to $a = \delta_{-1} - 2\delta_0 + \delta_1$ to obtain the following result.

Corollary

Let $0 < \alpha < 1$ *and* $f_{s,\alpha}$ *be the Lévy stable process. For* $n \in \mathbb{Z}$ *and* $0 < t < 1$, *we have*

$$\sum_{j=0}^{\infty} K_d^{\alpha j}(n) \frac{(-t)^j}{j!} = \int_0^{\infty} f_{t,\alpha}(s) e^{-2s} I_n(2s) ds;$$

in particular for $\alpha = \frac{1}{2}$,

$$\sum_{j=0}^{\infty} K_d^{\frac{j}{2}}(n) \frac{(-t)^j}{j!} = \int_0^{\infty} \frac{t}{\sqrt{4\pi s^3}} e^{\frac{-t^2}{4s}} e^{-2s} I_n(2s) ds.$$

5. Fundamental solutions for discrete evolution equations

We consider the fractional differential equation

$$\begin{cases} \mathbb{D}_t^{\beta} u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0. \\ u(n,0) = \varphi(n), \ \ u_t(n,0) = \phi(n) & n \in \mathbb{Z}, \end{cases} \tag{5}$$

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$Bf(n) = (b * f)(n)$, with $b \in l^1(\mathbb{Z})$, $f \in l^p(\mathbb{Z})$, $p \in [1, \infty]$ and $\beta \in (0, 2]$ and

$$\mathbb{D}_t^{\beta} v(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} v'(s) ds, \qquad t > 0,$$

for $0 < \beta < 1$ and

$$\mathbb{D}_t^{\beta} v(t) = \frac{1}{\Gamma(2-\beta)} \int_0^t (t-s)^{1-\beta} v''(s) ds, \qquad t > 0,$$

for $1 < \beta < 2$.

5. Fundamental solutions for discrete evolution equations

Theorem

Let $\varphi, \phi \in \ell^p(\mathbb{Z})$, *and* $g : \mathbb{Z} \times \mathbb{R}_+ \to \mathbb{C}$ *be such that, for each* $t \in \mathbb{R}_+$, $g(\cdot, t) \in \ell^p(\mathbb{Z})$ *and* $\sup_{s\in[0,t]} ||g(\cdot, s)||_p < \infty$ *with* $1 \le p \le \infty$.

(i) *For* $0 < \beta < 1$, *the function*

$$\begin{aligned} u(n,t) =& (E_{\beta,1}(t^\beta b) * \varphi)(n) \\ &+ \int_0^t (t-s)^{\beta-1} \left(E_{\beta,\beta}((t-s)^\beta b) * g(\cdot,s) \right)(n) ds, \;\; n \in \mathbb{Z}, \end{aligned}$$

is the unique solution of the initial value problem and $u(\cdot, t)$ *belong to* $\ell^p(\mathbb{Z})$ *for* $t > 0$.

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(ii) *For $1 < \beta < 2$, the function*

$$u(n,t) = (E_{\beta,1}(t^\beta b) * \varphi)(n) + t(E_{\beta,2}(t^\beta b) * \phi)(n) + \int_0^t (t-s)^{\beta-1} \left(E_{\beta,\beta}((t-s)^\beta b) * g(\cdot, s) \right)(n) ds, \;\; n \in \mathbb{Z},$$

is the unique solution of the initial value problem and , $u(\cdot, t)$ belong to $\ell^p(\mathbb{Z})$ for $t > 0$.

5. Fundamental solutions for discrete evolution equations

Now we consider the behavior of the solution when $\beta \to 1, 2$. For simplicity, $g = 0$. When $\beta \to 1^-$, the solution of equation converges to semigroup family operators $E_{1,1}(tb)$, and for the case $\beta \to 2^-$, the solution of equation (1),

$$u(\cdot, t) = E_{\beta,1}(t^\beta b) * \varphi + tE_{\beta,2}(t^\beta b) * \phi, \qquad t > 0,$$

converges to unique mild solution of second order Cauchy problem. However, as in the scalar case, when $\beta \to 1^+$ the solution of the equation converges to

$$u(\cdot, t) = E_{1,1}(bt) + tE_{1,2}(tb), \qquad t > 0.$$

Note that this function is the solution of the following first order modified Cauchy problem

$$\begin{cases} v'(n,t) = Bv(n,t) + \phi(n), & n \in \mathbb{Z},\ t > 0, \\ v(n,0) = \varphi(n), & n \in \mathbb{Z}, \end{cases}$$

for $\phi, \varphi \in \ell^p(\mathbb{Z})$. This fact is in accordance with the interpolation property of the Caputo fractional derivative.

5. Fundamental solutions for discrete evolution equations

The fundamental solution $u_{\beta,1}$ are obtained by requiring by $\psi = \delta_0$ and $\phi = 0$. In the case $1 < \beta \leq 2$ (included the wave equation), a second fundamental solution $u_{\beta,2}$ is given by $\psi = 0$ and $\phi = \delta_0$.

Corollary

Let $u_{\beta,1}$ and $u_{\beta,2}$ be the fundamental solutions of problems (1) *and Φ_β the Wright function, $\Phi_\beta(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\beta n+1-\beta)}$.*

(i) *Let $0 < \beta < 1$. Then,*

$$u_{\beta,1}(n,t) = \int_0^\infty \Phi_\beta(\tau) u_{1,1}(n, \tau t^\beta) d\tau, \qquad n \in \mathbb{Z}, \ t > 0.$$

(ii) *Let $1 < \beta < 2$. Then*

$$\begin{aligned} u_{\beta,1}(n,t) &= \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau) u_{2,1}(n, \tau t^{\frac{\beta}{2}}) d\tau, , \qquad n \in \mathbb{Z}, \ t > 0, \\ u_{\beta,2}(n,t) &= \int_0^t \frac{(t-u)^{\frac{-\beta}{2}}}{\Gamma(1-\frac{\beta}{2})} \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau) u_{2,2}(n, \tau u^{\frac{\beta}{2}}) d\tau \, du. \end{aligned}$$

5. Fundamental solutions for discrete evolution equations

The particular case $B = -(-A)^{\alpha}$, where A is the infinitesimal generator of an uniformly bounded C_0-semigroup in $\mathcal{B}(\ell^p(\mathbb{Z}))$ has received a special attention, for example $B = -(-\Delta_d)^{\alpha}$. These proofs rest about the explicit expressions of $E_{\beta,1}(-t^{\beta} K_d^{\alpha})$, $E_{\beta,2}(-t^{\beta} K_d^{\alpha})$ and $E_{\beta,\beta}(-t^{\beta} K_d^{\alpha})$.

Corollary

Let $\varphi, \phi \in \ell^p(\mathbb{Z})$, and $g : \mathbb{Z} \times \mathbb{R}_+ \to \mathbb{C}$ be such that, for each $t \in \mathbb{R}_+$, $g(\cdot, t) \in \ell^p(\mathbb{Z})$ and $\sup_{s \in [0,t]} ||g(\cdot, s)||_p < \infty$ with $1 \leq p \leq \infty$. Take $a \in \ell^1(\mathbb{Z})$ such that generates a uniformly continuous semigroup in $\ell^1(\mathbb{Z})$, we write $(-a)^{\alpha}$ the fractional powers given and $B(f) := -(-a)^{\alpha} * f$ for $f \in \ell^p(\mathbb{Z})$ and $0 < \alpha < 1$. Then the same representation of the fundamental solutions with $b = -(-a)^{\alpha}$ holds.

6. Applications to concrete examples

6.1 The discrete Nagumo equation Let us consider the linear part of the discrete Nagumo equation, which can be written as follows:

$$\begin{cases} \partial_t u(n,t) = \Delta_d u(n,t) - ku(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}. \end{cases} \tag{6}$$

where $0 < k < 1/2$. The discrete Nagumo equation is used as a model for the spread of genetic traits and for the propagation of nerve pulses in a nerve axon, neglecting recovery. Then

$$\sigma(e^{t(\Delta_d - kI)}) = e^{t\sigma(\Delta_d - kI)} = \{e^{ts} : t \geq 0, -4 - k \leq s \leq -k\}$$

It implies that the unique solution of equation (6) is uniformly asymptotically stable, i.e.

$$u(n,t) = e^{t(\Delta_d - kI)}\varphi(n) \to 0 \text{ as } t \to \infty.$$

6. Applications to concrete examples

6.1 The discrete Nagumo equation

Moreover, using Theorem 6(4) and the semigroup property, we can obtain a representation of the fundamental solution as follows:

$$u(n,t) = e^{-2t} \sum_{j=0}^{n} \sum_{l=0}^{n-j} \frac{(-kt)^l}{l!} I_{n-j-l}(2t)\varphi(j).$$

Since $\sigma(-(-\Delta_d)^\alpha) = [-4^\alpha, 0]$ we have that the same asymptotic behavior also holds for the fundamental solution of the fractional Laplacian version for the discrete Nagumo equation [LR]:

$$\begin{cases} \partial_t u(n,t) = -(-\Delta_d)^\alpha u(n,t) - ku(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}. \end{cases}$$

7. Applications to concrete examples

7.3 Subordination principle on Wright function We obtain some known formulae but others seem to be new.
Take $a = \delta_{-1} - \delta_0$ or $a = \delta_1 - \delta_0$.

(i) For $0 < \beta < 1$, $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have

$$E_{\beta,1}^{(n)}(t) = \sum_{j=0}^{\infty} \frac{(j+n)!}{j!} \frac{t^j}{\Gamma(\beta(j+n)+1)} = \int_0^{\infty} \Phi_\beta(\tau) e^{\tau t} \tau^n d\tau.$$

(ii) For $1 < \beta < 2$, $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have

$$(2t)^{n-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(-1)^j (j+n)!}{j!} \frac{t^{2j}}{\Gamma(\beta(j+n)+1)}$$

$$= \frac{\sqrt{\pi}}{2} \int_0^{\infty} \Phi_{\frac{\beta}{2}}(\tau) \tau^{n+\frac{1}{2}} J_{n-\frac{1}{2}}(\tau t)\ d\tau.$$

7. Applications to concrete examples

7.3 Subordination principle on Wright function

Now take $a = \delta_{-1} - 2\delta_0 + \delta_1$ or $a = \delta_{-2} - 2\delta_0 + \delta_2$.

(i) For $0 < \beta < 1$, $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have

$$\sum_{j=0}^{\infty} (-1)^j \binom{2(j+n)}{j} \frac{t^{j+n}}{\Gamma(\beta(j+n)+1)} = \int_0^{\infty} \Phi_\beta(\tau) e^{-2\tau t} I_n(2\tau t) d\tau.$$

In particular, when $\beta = \frac{1}{3}$, we get the integral formula for Airy function,

$$\sum_{j=0}^{\infty} (-1)^j \binom{2(j+n)}{j} \frac{t^{j+n}}{\Gamma(\frac{j+n}{3}+1)} = \int_0^{\infty} 3^{\frac{2}{3}} Ai\left(\frac{\tau}{3^{\frac{1}{3}}}\right) e^{-2\tau t} I_n(2\tau t) d\tau,$$

for $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$.

(ii) For $1 < \beta < 2$, $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have

$$\sum_{j=0}^{\infty} (-1)^j \binom{2(j+n)}{j} \frac{t^{2(j+n)}}{\Gamma(\beta(j+n)+1)} = \int_0^{\infty} \Phi_{\frac{\beta}{2}}(\tau) J_{2n}(2\tau t) d\tau.$$

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GRAFFITIS MATEMATICOS: Antonio Ledesma López

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