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CONSTRUCTIVE METHOD TO SOLVING 3D NAVIER – STOKES EQUATIONS

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Abstract

We consider 3D Navier – Stokes equations for motiom of incompressible medium and set ourselves the goal of creating a constructive solution method taking into account of all nonlinear terms. We proposes an approach to this problem the essence of which is to reduce the basic problem to a set of simple tasks.

1. The Navier — Stokes Equations. Navier — Stokes equations describe fluid and gases medium motion in presence of viscosity. Equation of that type are of mathematical interest and have a lot of applications to practical problems [1-2].

For 3D motion of a viscous incompressible medium the Navier - Stokes equations in dimensionless variables have the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial(p+\Phi)}{\partial x} + \frac{1}{Re}\cdot\Delta u, \qquad (1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial(p+\Phi)}{\partial y}+\frac{1}{Re}\cdot\Delta v, \quad (2)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial(p+\Phi)}{\partial z}+\frac{1}{Re}\cdot\Delta w, \quad (3)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \quad (4)$$

On equations (1-4) the main unknowns are the components of the velocity vector u, v, w and pressure p;

Δ is a three-dimentional Laplace operator on spatial coordinates,

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2};$$

Φ is the popential of external force;

Re is a positive parameter named the Reynolds number.

2. Approach to problem solution. The essence of the proposed approach is to reduce the basic problem of solution of the initial equations to a set of simple tasks. We face to five more simple tasks that should be consistently resolved. They are as follows [3-4].

2.1. Free divergence form.

Each of the separate Navier — Stokes equations, including the continuity one can be represented in free divergence form as

$$\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}+\frac{\partial R_i}{\partial z}+\frac{\partial S_i}{\partial t}=0. \quad (5)$$

Where P_i, Q_i, R_i, S_i, are some combinations of main unknowns u, v, w, p, and first derivatives by coordinates. Every equation of the form (5) allows integration in general

$$P_i=\frac{\partial\Psi_{2,i}}{\partial y}-\frac{\partial\Psi_{4,i}}{\partial z}-\frac{\partial\Psi_{6,i}}{\partial t}+\alpha_i,\ \ Q_i=-\frac{\partial\Psi_{2,i}}{\partial x}+\frac{\partial\Psi_{5,i}}{\partial z}-\frac{\partial\Psi_{3,i}}{\partial t}+\beta_i,$$

$$R_i=\frac{\partial\Psi_{4,i}}{\partial x}-\frac{\partial\Psi_{5,i}}{\partial y}+\frac{\partial\Psi_{1,i}}{\partial t}+\gamma_i,\ \ S_i=-\frac{\partial\Psi_{6,i}}{\partial x}+\frac{\partial\Psi_{3,i}}{\partial y}-\frac{\partial\Psi_{1,i}}{\partial t}+\delta_i. \quad (6)$$

Where $\Psi_{k,i}$, $k=1,2,...,6$ are some twice differentiable functions in four variables, α_i, β_i, γ_i, δ_i are an arbitrary functions in three variables under conditions

$$\frac{\partial\alpha_i}{\partial x}=\frac{\partial\beta_i}{\partial y}=\frac{\partial\gamma_i}{\partial z}=\frac{\partial\delta_i}{\partial t}=0.$$

While $3D$ Navier — Stokes equations including the continuity one combine four ratios so we have relations as (6) for everyone of $i=1,2,3,4$. In total, we have 16 equations of the form (6)

2.2. Integral of the Navier-Stokes equations.
Equality of the form (6) can be converted so as to exclude any nonlinear and non-divergent terms. As the result we arrive to nine equations linked main unknown u, v, w, p, associated ones $\Psi_i, (i = 1, 2, ..., 15)$, and an arbitrary additive functions in three variables α_i, β_i, γ_i, δ_i

$$p - p_0 + \Phi + \frac{U^2}{2} + d + d_t = \alpha_4 + \beta_4 + \gamma_4, \tag{7}$$

$$u^2 - v^2 + \frac{2}{Re}(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = -\frac{\partial^2 \Psi_{10}}{\partial x^2} + \frac{\partial^2 \Psi_{10}}{\partial y^2} - \frac{\partial^2 \Psi_{11}}{\partial z^2} - \frac{\partial^2 \Psi_{12}}{\partial z^2} + \frac{\partial^2 \Psi_{15}}{\partial y \partial z} +$$
$$\frac{\partial^2 \Psi_{14}}{\partial x \partial z} + \frac{\partial}{\partial t}(-\frac{\partial \Psi_1}{\partial x} + \frac{\partial \Psi_3}{\partial y} + \frac{\partial(\Psi_5 + \Psi_6)}{\partial z}) + 3(\alpha_4 - \beta_4), \tag{8}$$

$$v^2 - w^2 + \frac{2}{Re}(-\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = \frac{\partial^2 \Psi_{10}}{\partial x^2} + \frac{\partial^2 \Psi_{11}}{\partial x^2} - \frac{\partial^2 \Psi_{12}}{\partial y^2} + \frac{\partial^2 \Psi_{12}}{\partial z^2} - \frac{\partial^2 \Psi_{13}}{\partial x \partial y} -$$
$$\frac{\partial^2 \Psi_{14}}{\partial x \partial z} + \frac{\partial}{\partial t}(\frac{\partial(\Psi_1 + \Psi_2)}{\partial x} + \frac{\partial \Psi_4}{\partial y} - \frac{\partial \Psi_6}{\partial z}) + 3(\beta_4 - \gamma_4), \tag{9}$$

$$uv - \frac{1}{Re}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) = -\frac{\partial^2 \Psi_{10}}{\partial x \partial y} + \frac{1}{2}\frac{\partial}{\partial z}(-\frac{\partial \Psi_{15}}{\partial x} + \frac{\partial \Psi_{14}}{\partial y} + \frac{\partial \Psi_{13}}{\partial z}) +$$
$$\frac{1}{2}\frac{\partial}{\partial t}(-\frac{\partial \Psi_3}{\partial x} - \frac{\partial \Psi_1}{\partial y} - \frac{\partial(\Psi_8 + \Psi_9)}{\partial z}) + \frac{1}{2}(\frac{\partial \alpha_1}{\partial z} - \frac{\partial \alpha_3}{\partial t} + \frac{\partial \beta_1}{\partial z} - \frac{\partial \beta_2}{\partial t}), \tag{10}$$

$$uw - \frac{1}{Re}(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}) = \frac{\partial^2 \Psi_{11}}{\partial x \partial z} + \frac{1}{2}\frac{\partial}{\partial y}(-\frac{\partial \Psi_{15}}{\partial x} - \frac{\partial \Psi_{14}}{\partial y} - \frac{\partial \Psi_{13}}{\partial z}) +$$
$$\frac{1}{2}\frac{\partial}{\partial t}(-\frac{\partial \Psi_5}{\partial x} + \frac{\partial(\Psi_9 - \Psi_7)}{\partial y} + \frac{\partial \Psi_2}{\partial z}) - \frac{1}{2}(\frac{\partial \alpha_1}{\partial y} + \frac{\partial \alpha_2}{\partial t} - \frac{\partial \gamma_1}{\partial y} + \frac{\partial \gamma_3}{\partial t}), \tag{11}$$

$$vw - \frac{1}{Re}(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}) = -\frac{\partial^2 \Psi_{12}}{\partial y \partial z} + \frac{1}{2}\frac{\partial}{\partial x}(\frac{\partial \Psi_{14}}{\partial y} + \frac{\partial \Psi_{15}}{\partial x} - \frac{\partial \Psi_{13}}{\partial z}) +$$
$$\frac{1}{2}\frac{\partial}{\partial t}(\frac{(\partial \Psi_7 + \Psi_8)}{\partial x} + \frac{\partial \Psi_6}{\partial y} + \frac{\partial \Psi_4}{\partial z}) - \frac{1}{2}(\frac{\partial \beta_1}{\partial x} + \frac{\partial \beta_3}{\partial t} + \frac{\partial \gamma_1}{\partial x} + \frac{\partial \gamma_2}{\partial t}), \tag{12}$$

$$u = \frac{1}{2}(\frac{\partial}{\partial y}(-\frac{\partial \Psi_3}{\partial x} + \frac{\partial \Psi_1}{\partial y} + \frac{\partial \Psi_7}{\partial z}) + \frac{\partial}{\partial z}(-\frac{\partial \Psi_5}{\partial x} + \frac{\partial \Psi_8}{\partial y} - \frac{\partial \Psi_2}{\partial z})) +$$

$$\frac{1}{2}(\frac{\partial\alpha_2}{\partial z}+\frac{\partial\alpha_3}{\partial y}+\frac{\partial\delta_1}{\partial y}+\frac{\partial\delta_2}{\partial z}), \tag{13}$$

$$v=\frac{1}{2}(\frac{\partial}{\partial x}(\frac{\partial\Psi_3}{\partial x}-\frac{\partial\Psi_1}{\partial y}-\frac{\partial\Psi_7}{\partial z})+\frac{\partial}{\partial z}(\frac{\partial\Psi_9}{\partial x}+\frac{\partial\Psi_6}{\partial y}-\frac{\partial\Psi_4}{\partial z}))+$$
$$\frac{1}{2}(\frac{\partial\beta_2}{\partial x}+\frac{\partial\beta_3}{\partial z}-\frac{\partial\delta_1}{\partial x}+\frac{\partial\delta_3}{\partial z}), \tag{14}$$

$$w=\frac{1}{2}(\frac{\partial}{\partial x}(\frac{\partial\Psi_5}{\partial x}-\frac{\partial\Psi_8}{\partial y}+\frac{\partial\Psi_2}{\partial z})+\frac{\partial}{\partial y}(-\frac{\partial\Psi_9}{\partial x}-\frac{\partial\Psi_6}{\partial y}+\frac{\partial\Psi_4}{\partial z}))+$$
$$\frac{1}{2}(\frac{\partial\gamma_2}{\partial y}+\frac{\partial\gamma_3}{\partial x}-\frac{\partial\delta_2}{\partial x}-\frac{\partial\delta_3}{\partial y}). \tag{15}$$

The ratio (7) contains, on addition, values p_0, $\frac{U^2}{2}$, d and d_t. The first one is the additive pressure constant, the second one is the dimensionless velocity head

$$\frac{U^2}{2}=\frac{u^2+v^2+w^2}{2}.$$

Values d and d_t are dissipative terms defined by formulas

$$d=-\frac{U^2}{6}-\frac{1}{3}(\Delta_{xy}\Psi_{10}-\Delta_{xz}\Psi_{11}+\Delta_{yz}\Psi_{12}+\frac{\partial^2\Psi_{13}}{\partial x\partial y}-\frac{\partial^2\Psi_{14}}{\partial x\partial z}+\frac{\partial^2\Psi_{15}}{\partial y\partial z}), \tag{16}$$

$$d_t=\frac{1}{3}\frac{\partial}{\partial t}(\frac{\partial(\Psi_2-\Psi_1)}{\partial x}+\frac{\partial(\Psi_4-\Psi_3)}{\partial y}+\frac{\partial(\Psi_6-\Psi_5)}{\partial z}). \tag{17}$$

Symbols Δ_{yz}, Δ_{xz}, Δ_{xy} in (16) denotes the incomplete Laplace operators with respect to spatial coordinates

$$\Delta_{yz}=\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},\quad \Delta_{xz}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2},\quad \Delta_{xy}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

For associated unknown compared to (6) introduced more simple designation and proposed the name as stream pseudo function. Considered together these nine ratios provide the first integral of 3D Navier — Stokes equations.

Of the 9 ratios (7-15), expressions (8-12) are especially emphasized, since they represent a general structure for the main unknowns u, v, w, p.

2.3. Generator of solutions. Of the nine received five nonlinear relations. They contain quadratic nonlinear terms of Riccati's type. These nonlinear equations can be resolved relative to six unknown $\Psi_j, j = 10, 11, ..., 15$ only if two conditions of compatibility are fulfilled. They reduce to two equations of the fifth order with respect to nine associated unknown $\Psi_n, n = 1, 2, ..., 9$

$$\frac{\partial^2 f_2}{\partial x \partial y} - \frac{\partial^2 f_4}{\partial x^2} + \frac{\partial^2 f_4}{\partial y^2} + \frac{\partial^2 f_5}{\partial y \partial z} - \frac{\partial^2 f_6}{\partial x \partial z} = 0,$$

$$\frac{\partial^2 f_3}{\partial y \partial z} + \frac{\partial^2 f_4}{\partial x \partial z} - \frac{\partial^2 f_5}{\partial x \partial y} - \frac{\partial^2 f_6}{\partial y^2} + \frac{\partial^2 f_6}{\partial z^2} = 0. \qquad (18)$$

where each of the function f_i $(i = 2, 3, 4, 5, 6)$ represents the sum of the terms of equations (8-12) respectively, not containing $\Psi_j, j = 10, 11, ..., 15$. These two equations represenst a system of two nonlinear equations with respect to nine unknowns $\Psi_k, k = 1, 2, ..., 9$. Each set of functions satisfying these equations leads to an exact solution of the Navier - Stokes equations (1-4). So, the equations (18) can be considered as the generator of solutions for $3D$ Navier — Stokes equations.

2.4. Determination of unknowns Ψ_j $(j = 10, 11, ..., 15.)$ To complete the solution remains to find unknown p. In order to find p you must found out six associated unknown $\Psi_j, j = 10, 11, ..., 15$. Three of them can be set arbitrary. These ones are $\Psi_{13}, \Psi_{14}, \Psi_{15}$. The remaining three are defined as solution of linear inhomogeneous equations

$$F_4 = -\frac{\partial^2 \Psi_{10}}{\partial x \partial y}, \quad F_5 = \frac{\partial^2 \Psi_{11}}{\partial x \partial z}, \quad F_6 = -\frac{\partial^2 \Psi_{12}}{\partial y \partial z}, \qquad (19)$$

where F_i are already known functions.

2.5. Determination of p.

All values presents in the structure formula (7) for unknown p are defined. Unknown p is easy to find. As a result all of the main unknown u, v, w, p, are found out. The solution of the Navier — Stokes equations is fully built.

Note that a similar approach can be applied to solving the 3D Euler equations for the motion of an incompressible medium. It is enough to put $\frac{1}{Re} = 0$ in all relations [5].

3. Results.

The method described above allows one to construct exact solutions of the 3D Navier - Stokes equations. One need to consistently determine all unknowns, starting with the associated ones. First, Ψ_i $(i = 1, 2, ..., 9)$ are determined, then u, v, w. Further Ψ_j $(j = 10, 11, ..., 15)$ and the last one is p. Thus, family of simpler tasks are sequentially resolved.

Let's give examples of some solutions obtained by this method [6].

3.1. Solution 1.

$$u = \frac{1}{2}(A_0\mu m_3 e^{\frac{N_1^2 t}{Re}+\mu(n_3x+m_3y+l_3z)} + B_0\xi l_3 e^{\frac{N_2^2 t}{Re}+\xi(n_3x+m_3y+l_3z)}),$$
$$v = \frac{1}{2}(-A_0\mu n_3 e^{\frac{N_1^2 t}{Re}+\mu(n_3x+m_3y+l_3z)} + C_0 l_3 e^{\frac{N_3^2 t}{Re}+(n_3x+m_3y+l_3z)}), \quad (20)$$
$$w = \frac{1}{2}(-B_0\xi n_3 e^{\frac{N_2^2 t}{Re}+\xi(n_3x+m_3y+l_3z)} - C_0 m_3 e^{\frac{N_3^2 t}{Re}+(n_3x+m_3y+l_3z)}).$$

$$p = p_0 - gz. \quad (21)$$

Where g is gravity, $N_3^2 = n_3^2 + m_3^2 + l_3^2$, $N_1 = \mu N_3$, $N_2 = \xi N_3$.

This formulas present a set of exact solutions of $3D$ Navier - Stokes equations and contain eight arbitrary constants n_3, m_3, l_3, μ, ξ, A_0, B_0, C_0.

3.2. Solution 2.

$$u = -e^{\frac{21t}{Re}}(A_0 e^{x-2y-4z} + B_0 e^{-x-4y-2z}),$$
$$v = \frac{1}{2}e^{\frac{21t}{Re}}(-A_0 e^{x-2y-4z} + 2C_0 e^{x+4y+2z})$$

,

$$w = \frac{1}{2}e^{\frac{21t}{Re}}(B_0 e^{-x-4y-2z} - 4C_0 e^{x+4y+2z}). \quad (22)$$

$$p - p_0 = -gz + e^{\frac{42t}{Re}}(3A_0C_0 e^{2x+2y-2z} + \frac{1}{4}A_0B_0 e^{-6y-6z}). \quad (23)$$

This formulas present a set of new exact solutions to 3D Navier — Stokes equations and contain three arbitrarily chosen constants A_0, B_0, C_0.

3.3. Solution 3.

$$u = -\frac{A_1 sh(\frac{Re\theta_1}{2}) - B_1 sin(\frac{Re\lambda_1}{2})}{2(cos^2(\frac{Re\lambda_1}{4}) + sh^2(\frac{Re\theta_1}{4}))} + \frac{B_3 sh(\frac{Re\theta_3}{2}) + A_3 sin(\frac{Re\lambda_3}{2})}{2(cos^2(\frac{Re\lambda_3}{4}) + sh^2(\frac{Re\theta_3}{4}))},$$

$$v=-\frac{A_2sh(\frac{Re\theta_2}{2})-B_2sin(\frac{Re\lambda_2}{2})}{2(cos^2(\frac{Re\lambda_2}{4})+sh^2(\frac{Re\theta_2}{4}))}+\frac{B_1sh(\frac{Re\theta_1}{2})+A_1sin(\frac{Re\lambda_1}{2})}{2(cos^2(\frac{Re\lambda_1}{4})+sh^2(\frac{Re\theta_1}{4}))},\quad (24)$$

$$w=-\frac{A_3sh(\frac{Re\theta_3}{2})-B_3sin(\frac{Re\theta_3}{2})}{2(cos^2(\frac{Re\lambda_3}{4})+sh^2(\frac{Re\theta_3}{4}))}+\frac{B_2sh(\frac{Re\theta_2}{2})+A_2sin(\frac{Re\lambda_2}{2})}{2(cos^2(\frac{Re\lambda_2}{4})+sh^2(\frac{Re\theta_2}{4}))},$$

$$p-p_0=-gz-\frac{u^2+v^2+w^2}{2}-\frac{\partial\varphi}{\partial t}.\quad (25)$$

Where notation is used

$$\varphi=\varphi_1+\varphi_2+\varphi_3,$$

$$\varphi_k=\frac{2}{Re}ln(cos^2\frac{Re\lambda_k}{4}+sh^2\frac{Re\theta_k}{4}),\ \ k=1,2,3.$$

$$\theta_1=A_1(x-x_0)-B_1(y-y_0),\ \ \lambda_1=B_1(x-x_0)+A_1(y-y_0).$$

$$\theta_2=A_2(y-y_0)-B_2(z-z_0),\ \ \lambda_2=B_2(y-y_0)+A_2(z-z_0),\quad (26)$$

$$\theta_3=A_3(z-z_0)-B_3(x-x_0),\ \ \lambda_3=B_3(z-z_0)+A_3(x-x_0).$$

Formulas (24-25) presents set of exact solutions to 3D Navier-Stokes equations whereas x_0,y_0,z_0 are arbitrary constants and $A_k(t),B_k(t)$ $k=1,2,3,$ are arbitrary functions of time satisfying the condition

$$A_1^2+A_2^2+A_3^2=B_1^2+B_2^2+B_3^2.$$

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