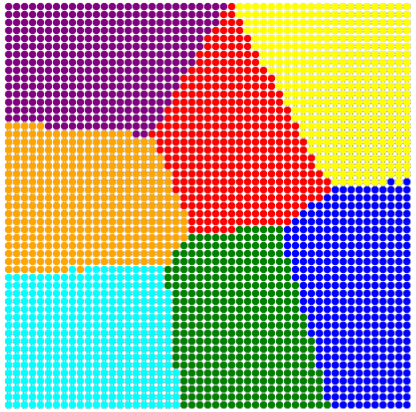
Optimal Grid Drawings of Complete Multipartite Graphs and an Integer Variant of the Algebraic Connectivity



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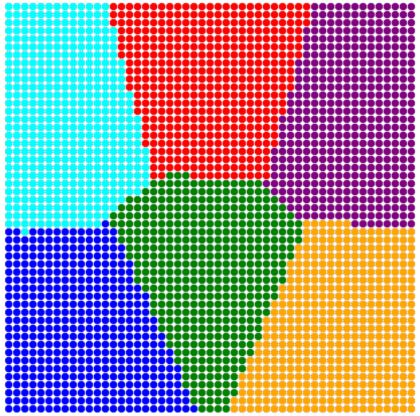

Joint work with

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Keywords:

algebraic connectivity

M. Fiedler

embedding lemma

D. A. Spielman, S.-H. Teng

minimum-2-sum-problem

M. Juvan, B. Mohar

centroidal Voronoi diagrams

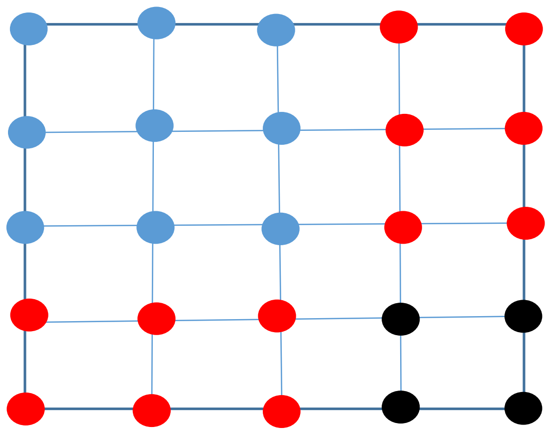
approximate eigenvectors with bounded integer coordinates

Problem 1

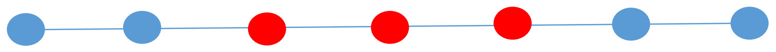
How to draw the vertices of a complete multipartite graph G on different points of a bounded d-dimensional integer grid, such that the sum of squared distances between vertices of G is minimized ? On each grid point one vertex of the graph is drawn.

Equivalent:

Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is minimized.



A drawing of the vertices of $K_{12,9,4}$ on a 2-dimensional grid.



A drawing of the vertices of $K_{3,4}$ on a line (d=1).

A relation between eigenvalues and drawings of a graph

Let $G = (V, E)$ be a graph and let $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_N(G)$ be the Laplacian eigenvalues of G.

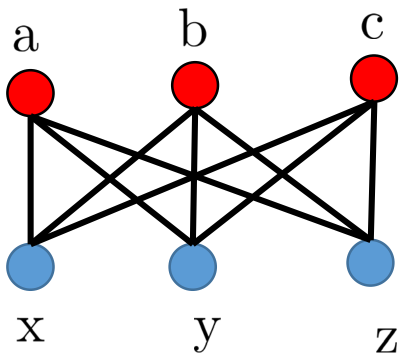
Embedding Lemma (Spielman-Teng):

$$\lambda_2(G) = \min \frac{\sum_{ij \in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i \in V} \|\vec{v}_i\|^2}$$

where the minimum is taken over all tuples $(\vec{v}_1, \ldots, \vec{v}_N)$ of vectors $\vec{v}_i \in \mathbb{R}^d$ with $\sum_{i=1}^{N} \vec{v}_i = \mathbf{0}$, and not all $\vec{v}_i$ are zero-vectors $\mathbf{0}$.

Spielman, D. A., Teng, S.-H. Spectral partitioning works: Planar graphs and finite element meshes. 2007.

Example (1):
An optimal drawing of the complete bipartite graph $K_{3,3}$



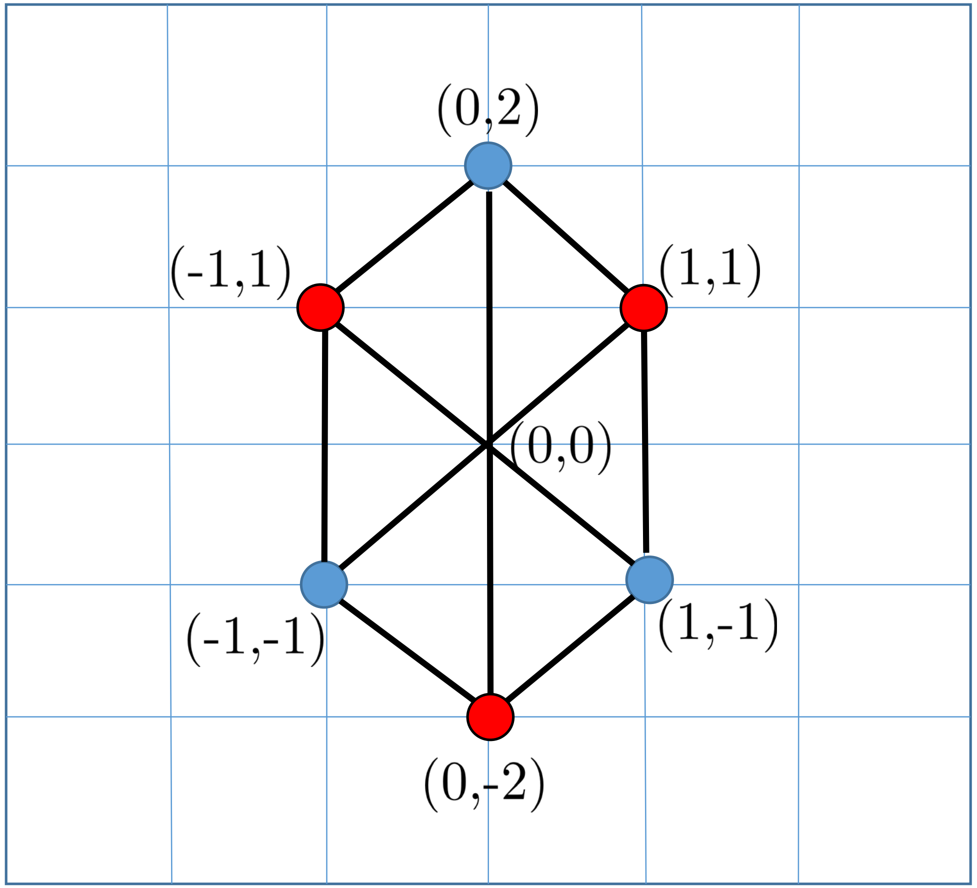
The complete bipartite graph $K_{3,3}$

$$\begin{array}{c} \\ a \\ b \\ c \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & x & y & z \end{array} \\ \begin{pmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ 0 & 3 & 0 & -1 & -1 & -1 \\ 0 & 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 & 0 \\ -1 & -1 & -1 & 0 & 3 & 0 \\ -1 & -1 & -1 & 0 & 0 & 3 \end{pmatrix} \end{array}$$

The Laplacian matrix of $K_{3,3}$

Laplacian eigenvalues $\{0^{(1)}, 3^{(4)}, 6^{(1)}\}$

Example (2):
An optimal drawing of the complete bipartite graph $K_{3,3}$



$$\lambda_2(G) = \min \frac{\sum_{ij \in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i \in V} \|\vec{v}_i\|^2}$$

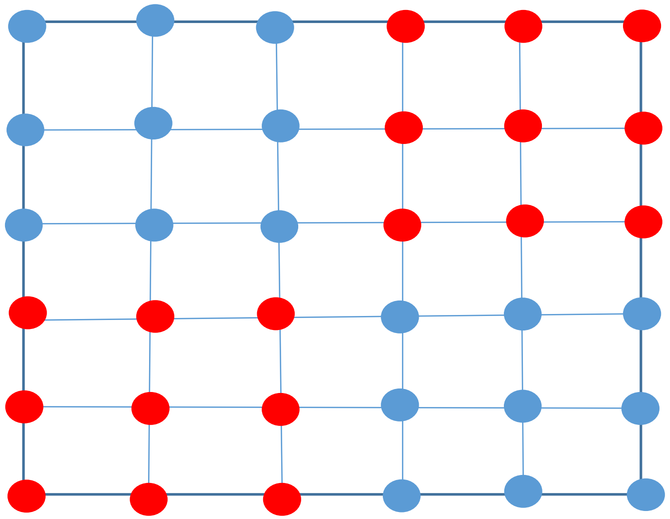
$$\lambda_2(G) \leq \frac{16 + 2 \cdot 8 + 2 \cdot 4 + 4 \cdot 2}{4 \cdot 2 + 2 \cdot 4} = 3$$

Laplacian eigenvalues $\{0^{(1)}, 3^{(4)}, 6^{(1)}\}$

Problem 1

Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is minimized.

If each color class C_i has the same number of points, then any drawing with $\sum_{\vec{v} \in C_i} \vec{v} = \mathbf{0}$, for all i, is optimal.



$$\lambda_2(G) = \min \frac{\sum_{ij \in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i \in V} \|\vec{v}_i\|^2}$$

There is an exponential number of optimal drawings.

Problem 1

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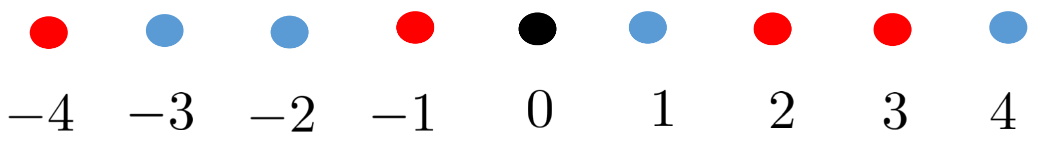
There is an exponential number of optimal drawings.

The number $\mathcal{N}$ of optimal drawings of $K_{1,2m,2m}$ for $d = 1$ is

$$c \cdot \frac{16^m}{m} < \mathcal{N} < 16^m.$$



Problem 1

Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is minimized.

Lemma:

For a straight line drawing of $G = (V, E) = K_{n_1,\dots,n_r}$ on "a small box of" the grid $\mathbb{Z}^d$,

$$\frac{\sum_{ij\in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i\in V} \|\vec{v}_i\|^2} = N + \frac{1}{S}\sum_{i=1}^{r}\left(-n_i \sum_{v\in C_i} \|v\|^2\right) + \frac{1}{S}\sum_{i=1}^{r}\left\|\sum_{v\in C_i} v\right\|^2$$

where

$N = n_1 + \ldots + n_r$ $\quad S = \sum_{i\in V} \|v_i\|^2$ $\quad$ color classes C_i

Problem 1

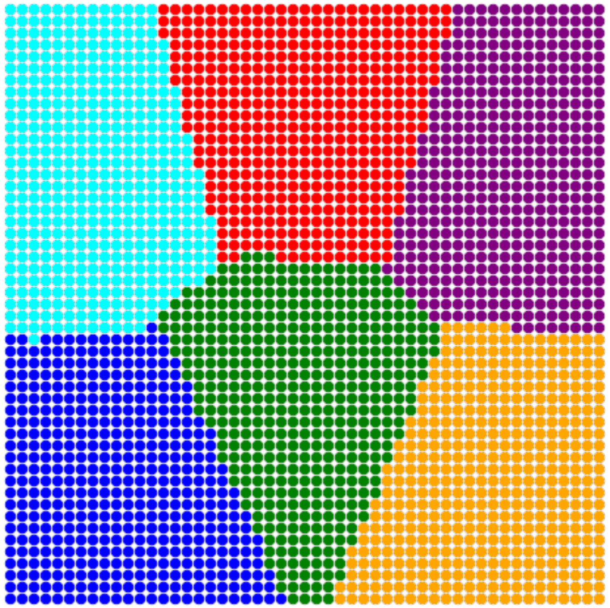
Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is minimized.

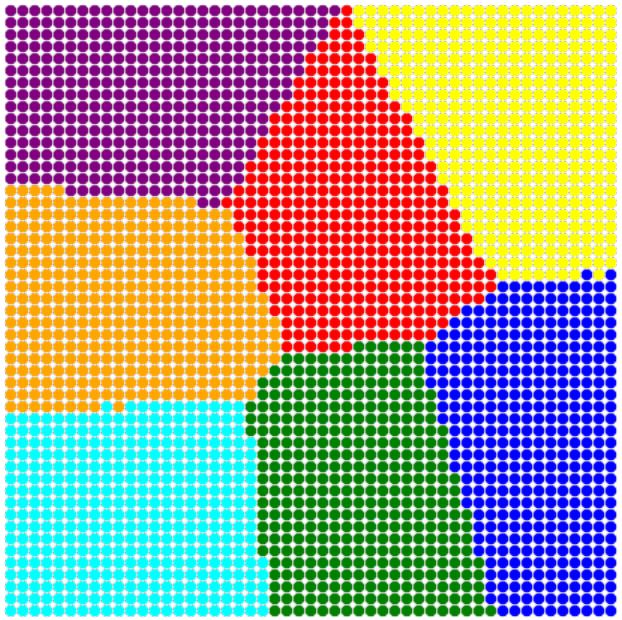
If all color classes C_i have a **different** number of points, then in an optimal drawing, for each i, the union of the smallest i color classes, $\bigcup_{j=1}^{i} C_j$, forms a ball centered at $\mathbf{0}$.

The optimal drawing is essentially unique.

Problem 2

Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is **maximized.**



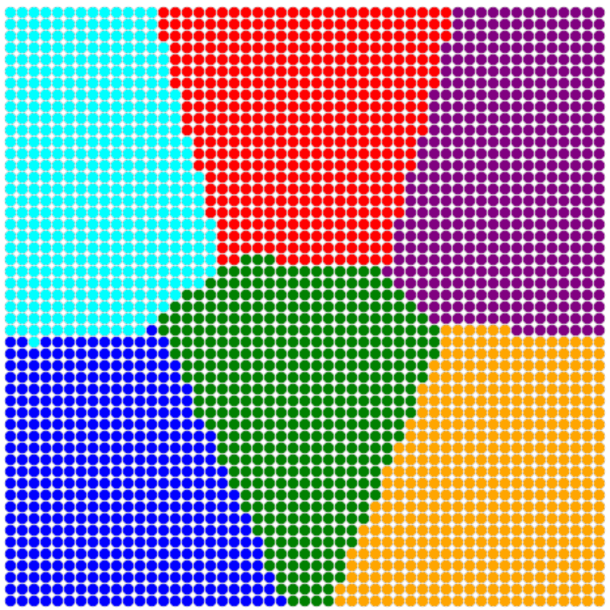


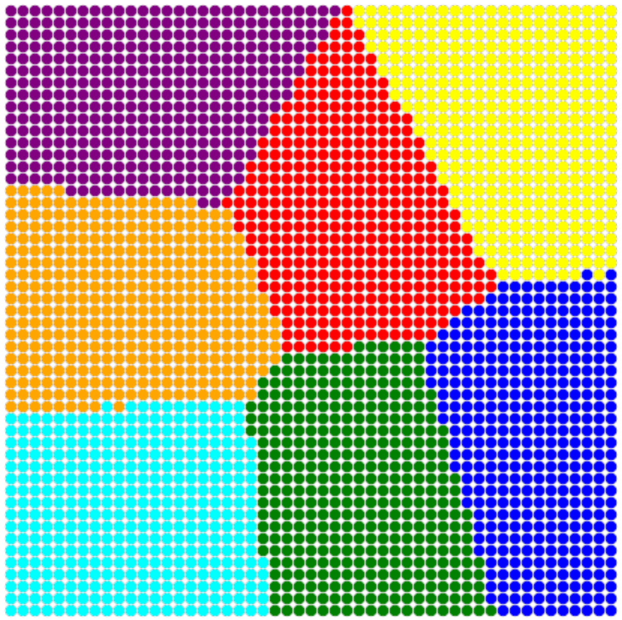
Drawings obtained with computer simulations, using simulated annealing.

The best way to maximize the sum of squared distances between points of different colors on a 101×101 integer grid. Left: for $r = 6$ colors, with 1701 purple points and 1700 points of every other color. Right: for $r = 7$ colors, with 1459 yellow points and 1457 points of every other color.

Problem 2

Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is **maximized.**





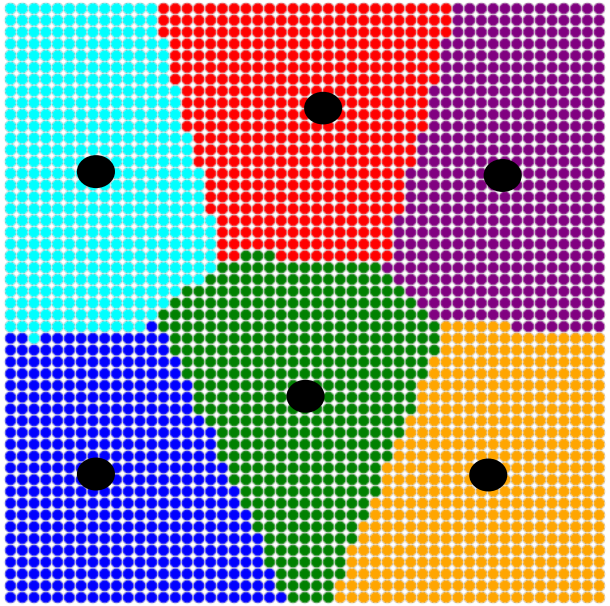
Embedding lemma for largest Laplacian eigenvalue:

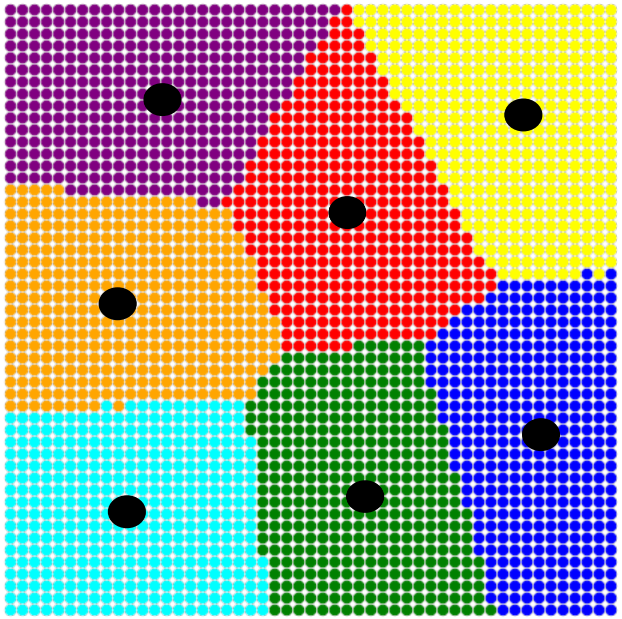
$$\lambda_N(G) = \max \frac{\sum_{ij \in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i \in V} \|\vec{v}_i\|^2}$$

Problem 2

Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is **maximized.**

centroidal Voronoi diagrams





Given a set of points $\{c_i\}_{i=1}^r$, the Voronoi region of a point c_i is defined by

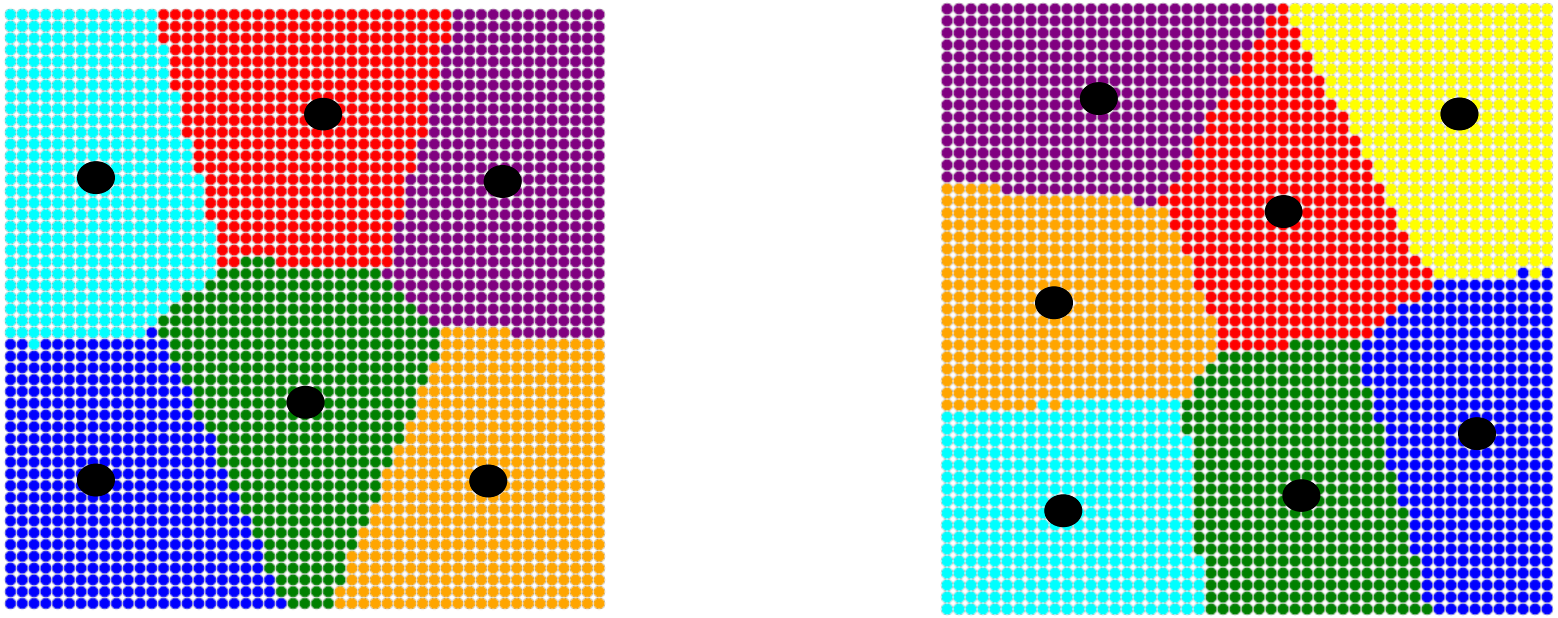
$$V_i = \{x \in \mathbb{R}^d \mid ||x - c_i|| < ||x - c_j|| \text{ for } j = 1, \ldots, r, j \neq i\}.$$

A Voronoi diagram is *centroidal* if each c_i is the center of mass of its Voronoi region.

Problem 2

Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is **maximized.**

centroidal Voronoi diagrams

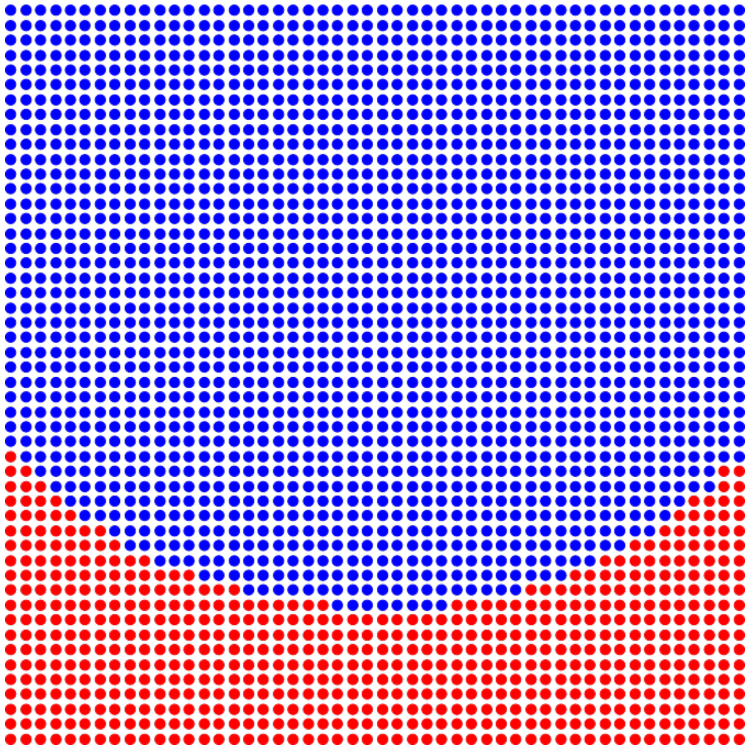


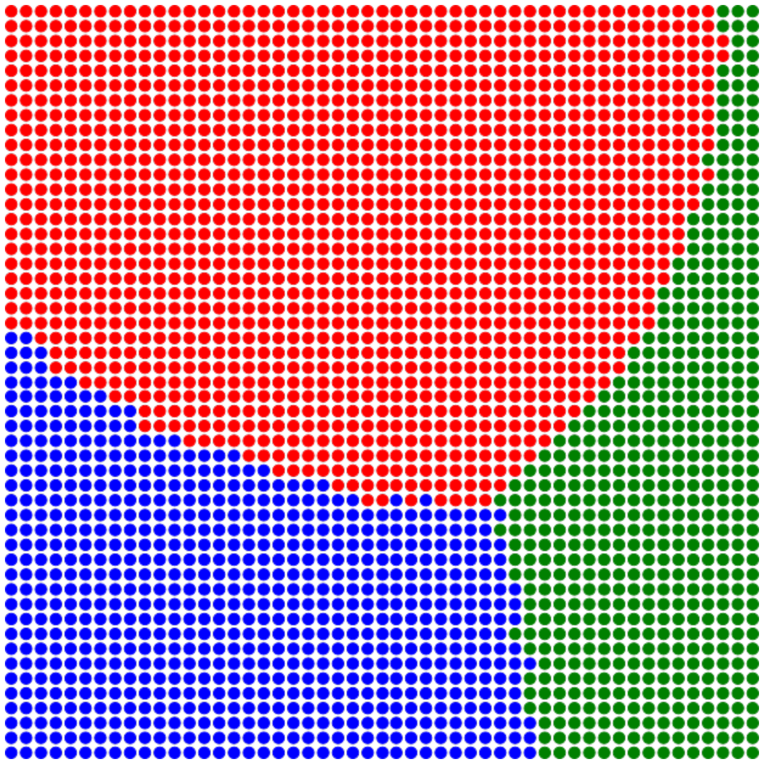
For a set of n points $v_1, \ldots, v_n$ in $\mathbb{R}^d$, with centroid $c = \frac{1}{n}\sum_{i=1}^{n} v_i$

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} ||v_i - v_j||^2 = n \sum_{i=1}^{n} ||v_i - c||^2$$

Problem 2

Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is **maximized.**





weighted centroidal Voronoi diagrams

if color classes of $K_{n_1,n_2,\ldots,n_r}$ have different sizes

Part 2

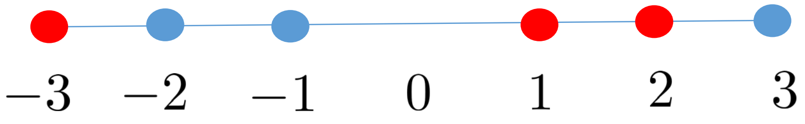
An Integer Variant of the Algebraic Connectivity

Consider drawings in dimension $d = 1$ of graphs $G = (V, E)$ with $|V| = N$.

Define $$\lambda_2^I(G) = \min \frac{\sum_{ij \in E} (v_i - v_j)^2}{\sum_{i \in V} v_i^2}$$

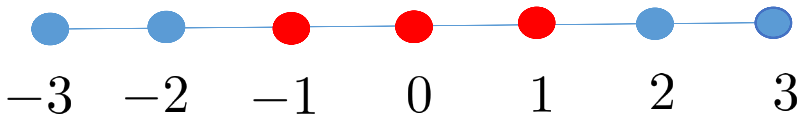
where the minimum is taken over drawings of G on $\{-\lfloor N/2 \rfloor, \ldots, \lfloor N/2 \rfloor\}$ with $\sum_{i=1}^{N} v_i = 0$ and no two vertices are mapped to the same point.

If N is even, then no vertex is mapped to the origin.



A drawing of vertices of $K_{3,3}$ on $\{-3, \ldots, 3\}$.

$\lambda_2^I(K_{3,3}) = \lambda_2(K_{3,3}) = 3$



A drawing of vertices of $K_{3,4}$ on $\{-3, \ldots, 3\}$.

$\lambda_2^I(K_{3,4}) > \lambda_2(K_{3,4}) = 3$

An Integer Variant of the Algebraic Connectivity

$$\lambda_2^I(G) = \min \frac{\sum_{ij \in E}(v_i - v_j)^2}{\sum_{i \in V} v_i^2}$$

where the minimum is taken over drawings of G on $\{-\lfloor N/2 \rfloor, \ldots, \lfloor N/2 \rfloor\}$ with $\sum_{i=1}^{N} v_i = 0$ and no two vertices are mapped to the same point.

For N odd, λ_2^I is equivalent to the **minimum-2-sum.**

The **minimum-p-sum-problem**: for $p > 0$, for a graph G and a bijective mapping Ψ from V to $\{1, \ldots, N\}$, define $\sigma_2(G, \Psi) = \left(\sum_{uv \in E(G)} |\Psi(u) - \Psi(v)|^p\right)^{1/p}$. The quantity $\sigma_p(G) = \min_\Psi \sigma_p(G, \Psi)$ (where the minimum is taken over all bijective mappings) is then called the minimum-p-sum of G. The problem of finding $\sigma_p(G)$ is called the minimum-p-sum problem.

Juvan, M., Mohar, B. Optimal linear labelings and eigenvalues of graphs. 1992.

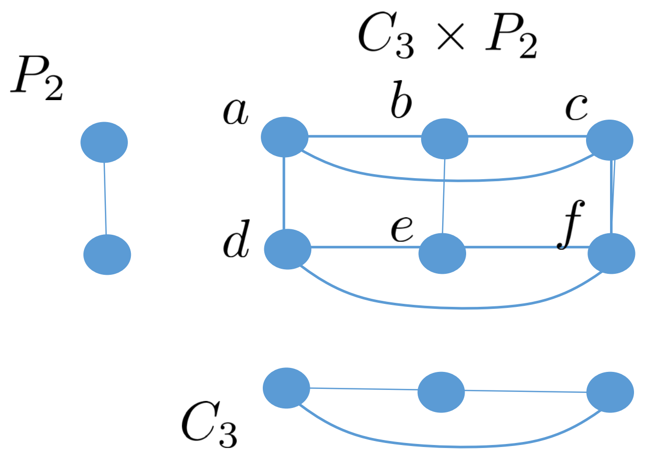
An Integer Variant of the Algebraic Connectivity

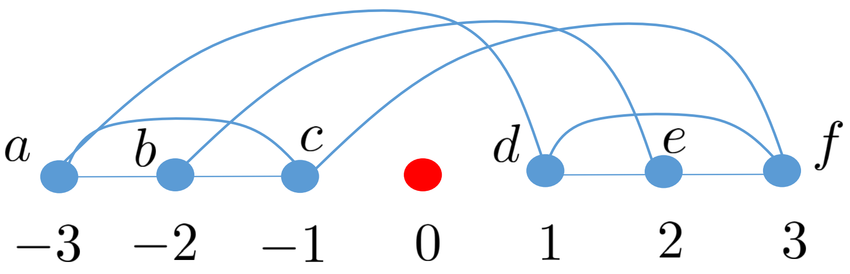
Properties of λ_2^I	Analogous properties of λ_2
If G and H are edge-disjoint graphs with the same set of vertices, then $\lambda_2^I(G) + \lambda_2^I(H) \leq \lambda_2^I(G \cup H)$	$\lambda_2(G) + \lambda_2(H) = \lambda_2(G \cup H)$
Denote by $G + e$ the graph obtained from the graph G with N vertices by adding an edge e. Then $\lambda_2^I(G) + \frac{1}{2\sum_{i=1}^{\lfloor N/2 \rfloor} i^2} \leq \lambda_2^I(G+e) \leq \lambda_2^I(G) + \frac{N^2}{2\sum_{i=1}^{\lfloor N/2 \rfloor} i^2}$	$\lambda_2(G) \leq \lambda_2(G+e) \leq \lambda_2(G) + 2$

The Cartesian product of graphs

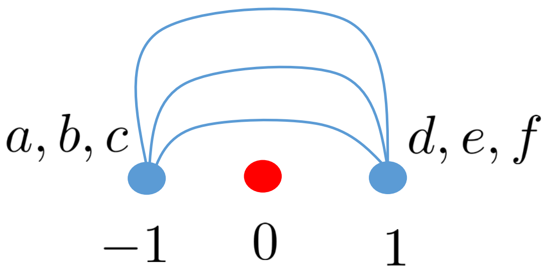
$\lambda_2^I(G \times H)$ can be strictly larger than $\min\{\lambda_2^I(G), \lambda_2^I(H)\}$

$\lambda_2(G \times H) = \min\{\lambda_2(G), \lambda_2(H)\}$





$$\lambda_2^I(C_3 \times P_2) = \frac{60}{28}$$



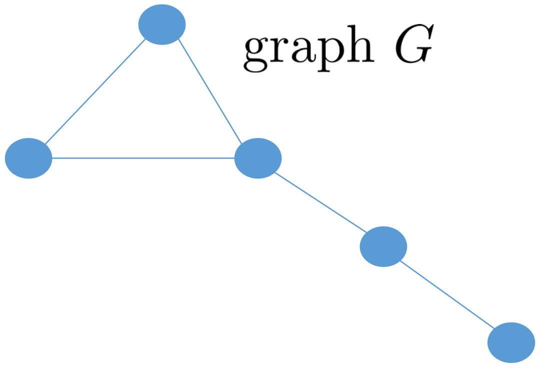
$$\lambda_2(C_3 \times P_2) = 2$$

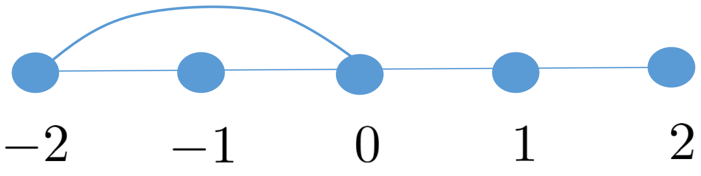
The Cartesian product of graphs

There are graphs G with $\lambda_2^I(G \times G) < \lambda_2^I(G)$.

$$\lambda_2(G \times G) = \lambda_2(G)$$

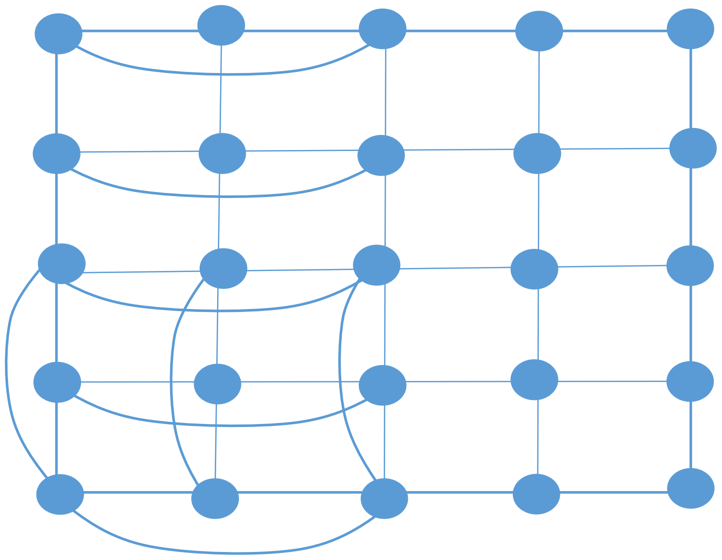
graph G





$$\lambda_2^I(G) = \frac{8}{10}$$

graph $G \times G$

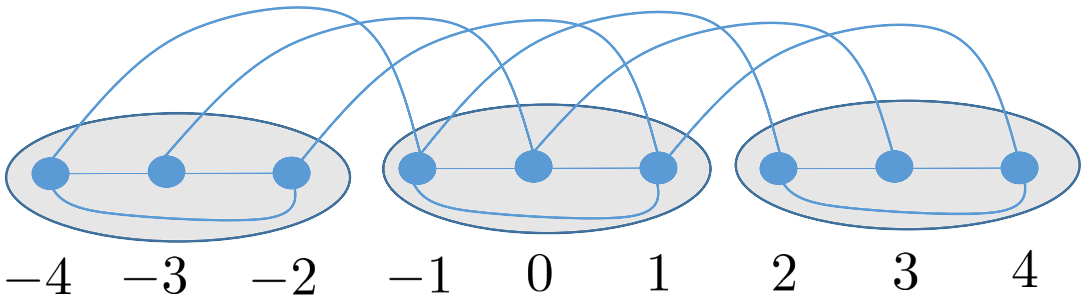


$$\lambda_2^I(G \times G) < \frac{6}{10}$$

The Cartesian product of graphs

Let G and H be graphs of odd order $|G|$ and $|H|$.

$$\lambda_2^I(G \times H) \leq \lambda_2^I(G)\left(\frac{|G|^2-1}{|G|^2|H|^2-1}\right) + \lambda_2^I(H)\left(\frac{|G|^2(|H|^2-1)}{|G|^2|H|^2-1}\right)$$



A drawing of $C_3 \times P_3$ that attains this bound.

The Cartesian product of graphs

Let G and H be graphs of odd order $|G|$ and $|H|$.

$$\lambda_2^I(G \times H) \leq \lambda_2^I(G)\left(\frac{|G|^2-1}{|G|^2|H|^2-1}\right) + \lambda_2^I(H)\left(\frac{|G|^2(|H|^2-1)}{|G|^2|H|^2-1}\right)$$

$$\lambda_2^I(G \times H) \leq \frac{\lambda_2^I(G) + \lambda_2^I(H)}{2}$$

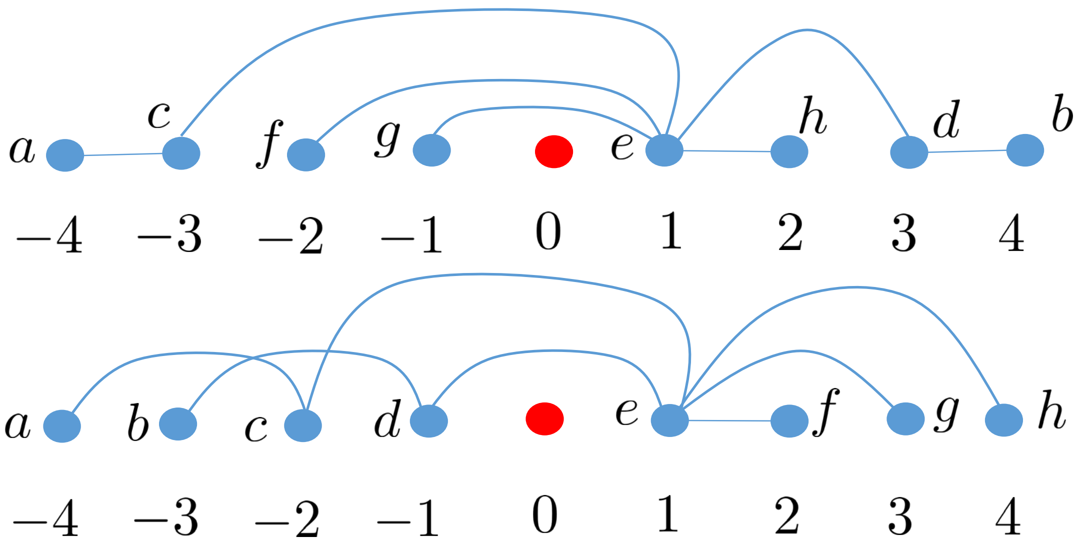
If $\lambda_2^I(G) = \lambda_2(G)$ and G has odd order, then $\lambda_2^I(G \times G) = \lambda_2^I(G)$.

Analogous property of λ_2

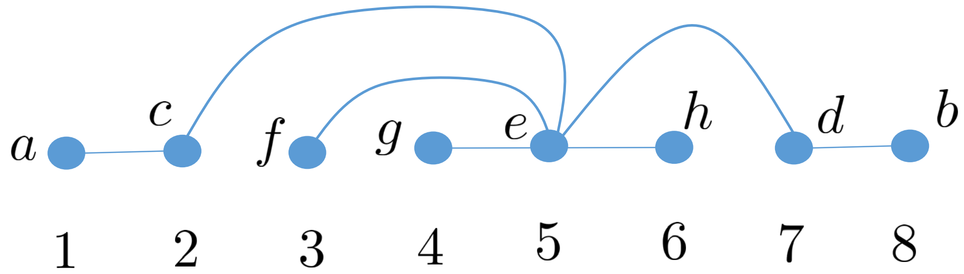
$$\lambda_2(G \times H) = \min\{\lambda_2(G), \lambda_2(H)\}$$

Comparing λ_2^I with the minimum 2-sum

There exist graphs G of even order, for which the optimal drawings of $\lambda_2^I(G)$ and $\sigma_2^2(G)$ are different.



the second drawing is better for $\lambda_2^I(G)$



optimal drawing for $\sigma_2^2(G)$

An Integer Variant of the Algebraic Connectivity

Open problem :
Characterize the class of graphs G for which $\lambda_2(G) = \lambda_2^I(G)$.

What we know:

$\lambda_2^I = \lambda_2$

- for the hypercube
- for complete multi-partite graphs $K_{n,n,\ldots,n}$ for $n > 1$

- If $\lambda_2^I(G) = \lambda_2(G)$ and $|G|$ is odd, then $\lambda_2^I(G \times G) = \lambda_2^I(G)$