Non-commutative polynomial optimization in quantum physics

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Computational aspects of commutative and noncommutative positive polynomials
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Polynomial optimization problems

$$\bar{p} = \min_{x_1,\ldots,x_n} p(x_1,\ldots,x_n)$$

$$\text{s.t.} \quad q_j(x_1,\ldots,x_n) \geq 0$$

where $x = (x_1,\ldots,x_n) \in R^n$

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Convergence can be proven using Putinar's result on Positivstellensatz.

Non-commutative polynomial optimization

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M. Navascués. S. Pironio, AA, Phys. Rev. Lett. 98, 010401 (2007); New J. Phys. 10 (7), 073013 (2008).

S. Pironio, M. Navascués, AA, SIAM J. Optim. 20, 2157 (2010).

Hierarchy of SDP relaxations

We consider the set of monomials of a given degree ≤ k on the previous operators, where each monomial is defined by a vector of indices α. The degree of the monomial is $|\alpha|=k$.

Example:

$$Y_{\alpha=(1,\bar{3},6,6)} = X_1 X_3^* X_6^2$$

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We consider maps from this set to complex numbers: $\Lambda\left(Y_\alpha\right) = y_\alpha \in C$

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Hierarchy of SDP relaxations

For a given sequence, we define the moment matrix M_k of degree k as

$$\left(M_k\right)_{\alpha,\beta} = \Lambda\left(X_{\bar{\alpha}} X_{\beta}\right)$$

$$|\alpha|,|\beta| \leq k$$

$$M_1 = \begin{pmatrix} 1 & y_1 & y_2 & y_{\bar{1}} & y_{\bar{2}} \\ y_{\bar{1}} & y_{\bar{1}1} & y_{\bar{1}2} & y_{\bar{1}\bar{1}} & y_{\bar{1}\bar{2}} \\ y_{\bar{2}} & y_{\bar{2}1} & y_{\bar{2}2} & y_{\bar{2}\bar{1}} & y_{\bar{2}\bar{2}} \\ y_1 & y_{11} & y_{12} & y_{1\bar{1}} & y_{1\bar{2}} \\ y_2 & y_{21} & y_{22} & y_{2\bar{1}} & y_{2\bar{2}} \end{pmatrix}$$

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For a polynomial $P(X) = \sum p_\delta X_\delta$ the localizing matrix $L_{P,k}$ of degree k is defined as

$$\left(L_{P,k}\right)_{\alpha,\beta} = \sum p_\delta \Lambda\left(X_{\bar{\alpha}} X_\delta X_\beta\right)$$

$$M_k = L_{1,k}$$

$$|\alpha|,|\beta| \le k$$

Hierarchy of SDP relaxations

Every specific choice of operators X_i and state ψ defines a map Λ:

$$\Lambda\left(X_\alpha\right) = \left\langle \psi \middle| X_\alpha \middle| \psi \right\rangle$$

For these maps, all localizing matrices defined for positive polynomials are positive.

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Proof:

$$\langle v|L_{P,k}|v\rangle=\sum_{\alpha,\beta}v_\alpha^*\left(L_{P,k}\right)_{\alpha,\beta}v_\beta=\sum_{\alpha,\beta}v_\alpha^*\sum_\delta p_\delta\Lambda\left(X_{\bar{\alpha}}X_\delta X_\beta\right)v_\beta=$$

$$\sum_{\alpha,\beta}v_\alpha^*\sum_\delta p_\delta\langle\psi|X_{\bar{\alpha}}X_\delta X_\beta|\psi\rangle v_\beta=\langle\psi|\left(\sum_\alpha v_\alpha^* X_{\bar{\alpha}}\right)\left(\sum_\delta p_\delta X_\delta\right)\left(\sum_\beta v_\beta X_\beta\right)|\psi\rangle\geq 0$$

Hierarchy of SDP relaxations

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Relaxation of order k:

$$p^{(k)} = \min \sum p_\delta y_\delta$$
$$\text{s.t.} \quad M_k \geq 0, L_{Q_j, k-\deg(Q_j)/2} \geq 0$$

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Clearly:

$$p^{(1)} \leq p^{(2)} \leq \ldots \leq \bar{p}$$

Convergence of the hierarchy

$$\bar{p} = \min \langle \psi | P(X_1, \ldots, X_n) | \psi \rangle$$
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Positivity domain: $S_Q = \{ (X_1, \ldots, X_n) \text{ s.t. } Q_j(X_1, \ldots, X_n) \geq 0 \}$

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If S_Q is bounded, we can make M_Q Archimedean by choosing a large enough C and adding to the set of polynomial constraints the condition:

$$C - X_1^* X_1 - \ldots - X_n^* X_n$$

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If M_Q is Archimedean: $$\lim_{k\to\infty} p^{(k)} = \overline{p}$$

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Convergence can be established at a finite step whenever the optimal solution y_k for relaxation of order k is such that:

$$\operatorname{rank}(M_k) = \operatorname{rank}\left(M_{k-\max(d_i)}\right)$$

Convergence of the hierarchy

Consider the problem:

$$\lambda^{(k)} = \max_{\lambda, B_i, C_{ij}} \lambda$$

$$\text{s.t. } P(X_1, \ldots, X_n) - \lambda = \sum_i B_i^* B_i + \sum_{i,j} C_{i,j}^* Q_j C_{i,j}$$

$$\max_i \deg(B_i) \leq k, \max_i \deg(C_{i,j}) \leq k - d_i$$

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This problem can be cast in a sdp form, and proven to be the dual of step *k* before.

$$\lambda^{(k)} \leq p^{(k)} \leq \overline{p}$$

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This defines a feasible point for the previous problem, so one has:

$$\overline{p}-\varepsilon \le \lambda^{(k)} \le p^{(k)} \le \overline{p} \quad \forall \varepsilon > 0$$

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Relation to classical SDP hierarchies

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⟷

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↓

$$p^{(1)} = \bar{p}$$ Easy!

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↓

NP-hard

Why do we care about this?

Quantum physics

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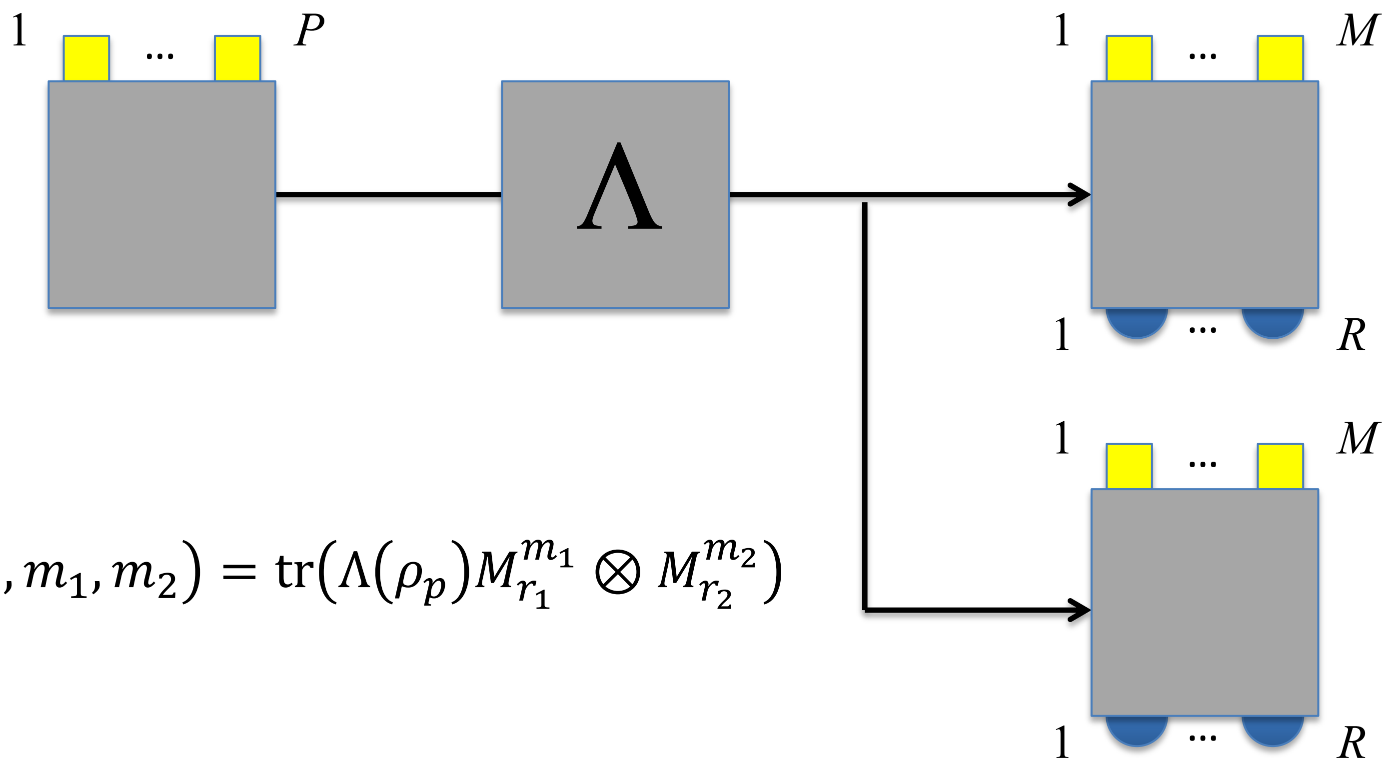
4. When combining two systems, A and B, with corresponding Hilbert spaces $\mathbb{C}^{d_A}$ and $\mathbb{C}^{d_B}$, the Hilbert space of the joint system is the tensor product of the two spaces.

Statistics in quantum experiments



$$P(r_1, r_2 | p, m_1, m_2) = \mathrm{tr}\left(\Lambda(\rho_p) M_{r_1}^{m_1} \otimes M_{r_2}^{m_2}\right)$$

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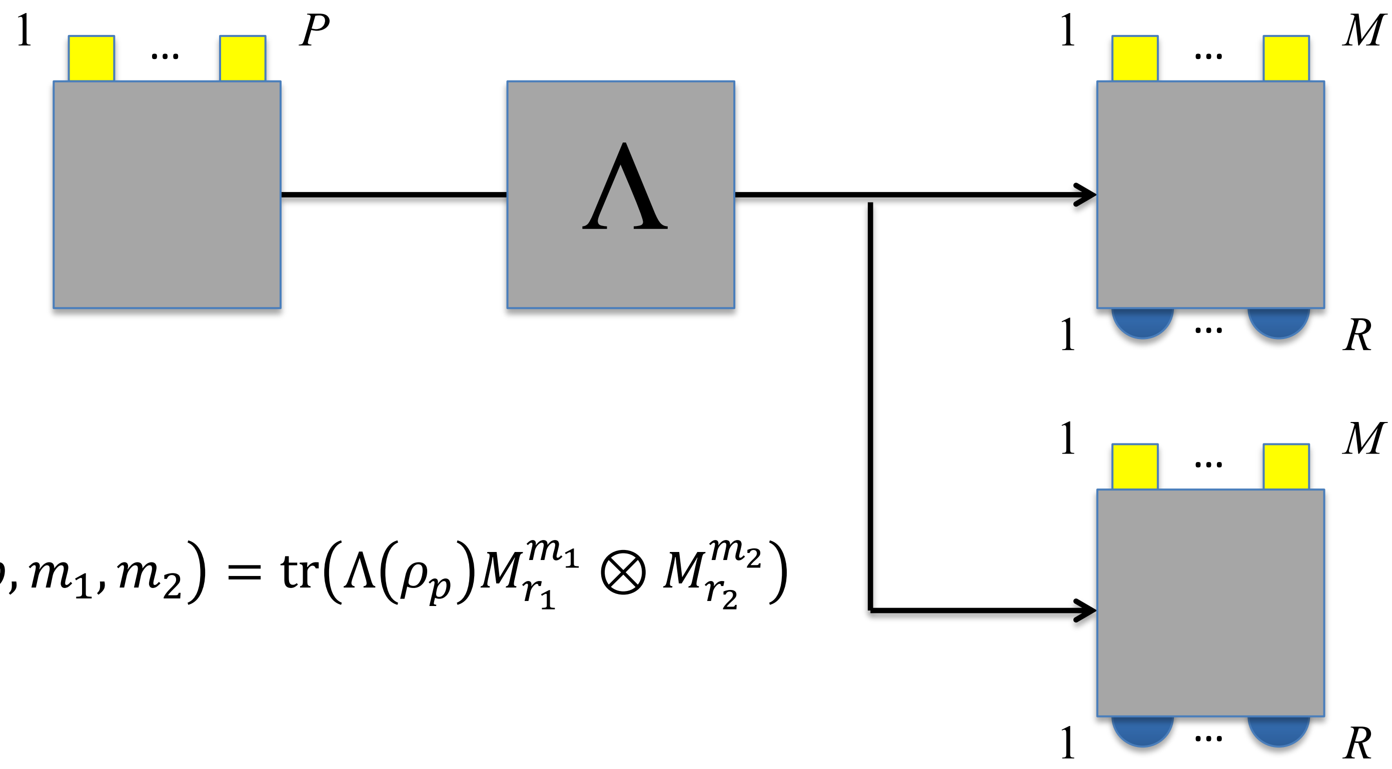

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Quantum physics is a natural source of problems involving polynomials of operators.

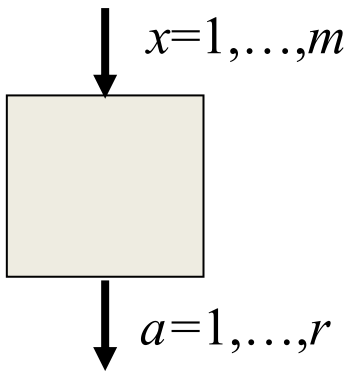
Characterization of Quantum Correlations

Navascués, Pironio, Acin, PRL 2007, NJP 2009

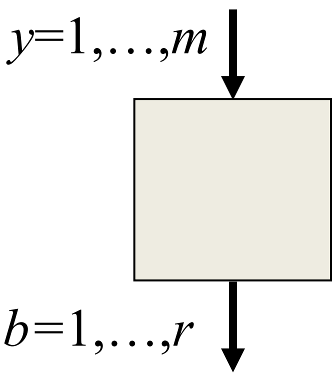
Physical correlations

The object we deal with is a conditional probability distribution of the outputs given the inputs, which encapsulates the correlations among devices.

Alice



$$p(a,b|x,y)$$



Bob

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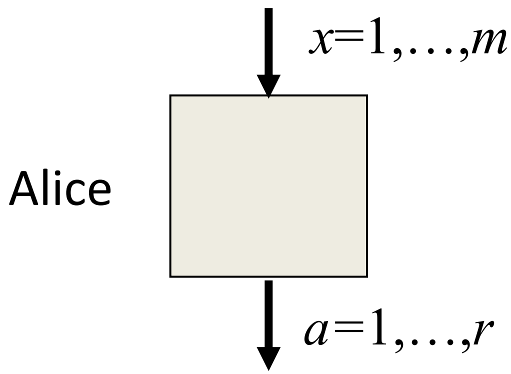

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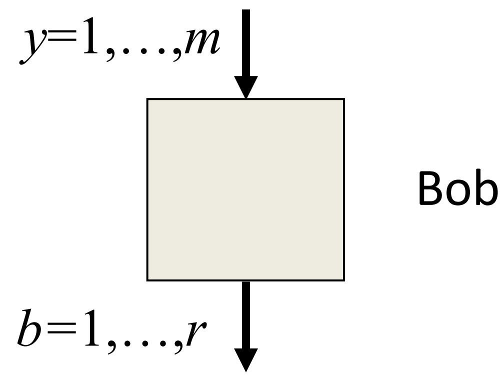

$$p(a,b|x,y) = \begin{pmatrix} p(1,1|1,1) & p(1,2|1,1) & \cdots & p(r,r|1,1) \\ & & & \\ & & & \end{pmatrix} \qquad \sum_{a,b} p(a,b|1,1) = 1$$

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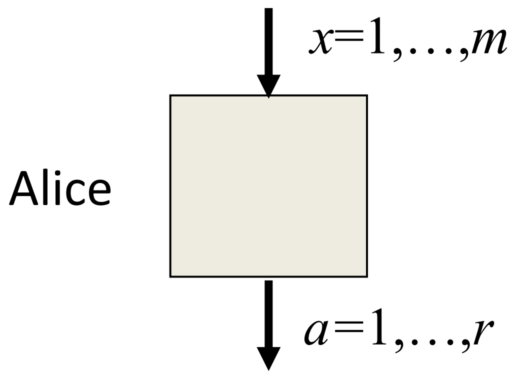

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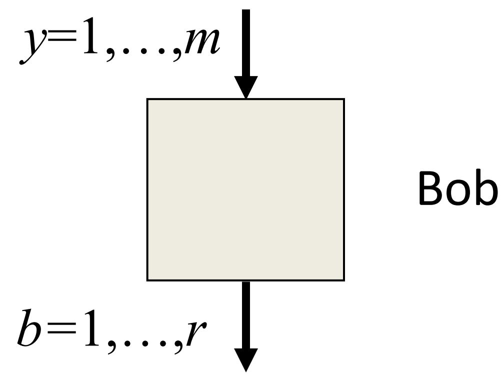

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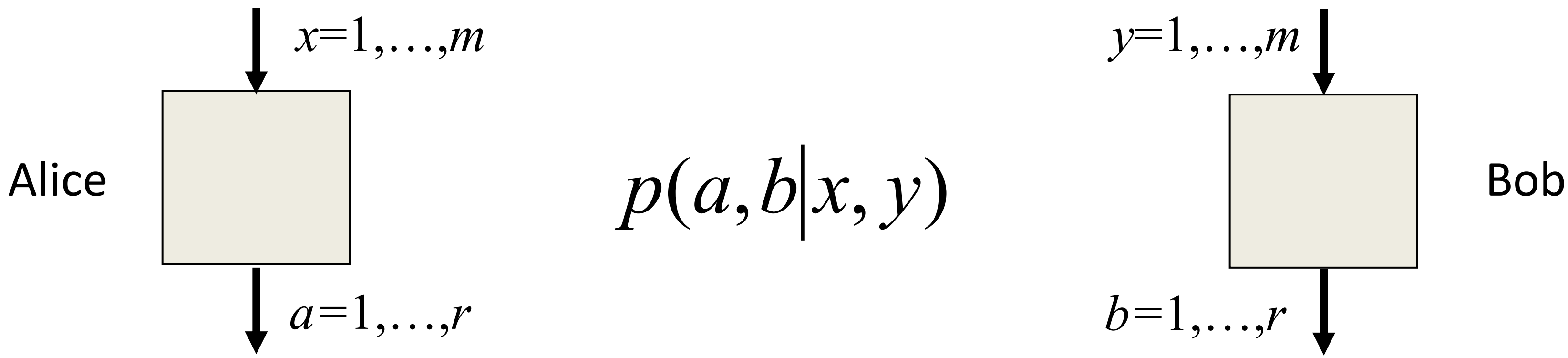

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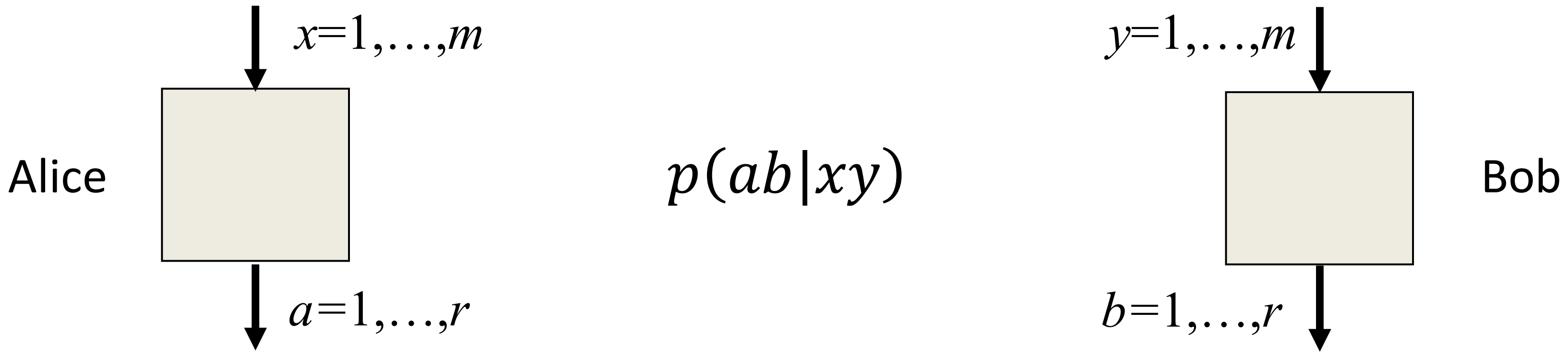
$$p(a,b|x,y)\in\Re^{m^2r^2}$$

$$p(a,b|x,y)\geq 0$$

$$\sum_{a,b}p(a,b|x,y)=1$$

Example



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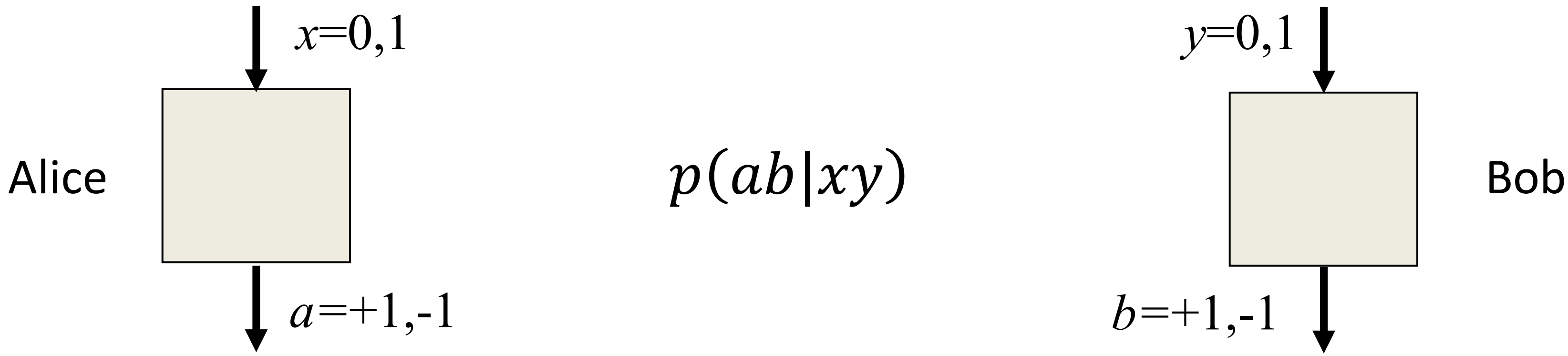

$$p(ab|xy) = \begin{pmatrix} p(+1,+1|0,0) & p(+1,-1|0,0) & p(-1,+1|0,0) & p(-1,-1|0,0) \\ p(+1,+1|0,1) & p(+1,-1|0,1) & p(-1,+1|0,1) & p(-1,-1|0,1) \\ p(+1,+1|1,0) & p(+1,-1|1,0) & p(-1,+1|1,0) & p(-1,-1|1,0) \\ p(+1,+1|1,1) & p(+1,-1|1,1) & p(-1,+1|1,1) & p(-1,-1|1,1) \end{pmatrix}$$

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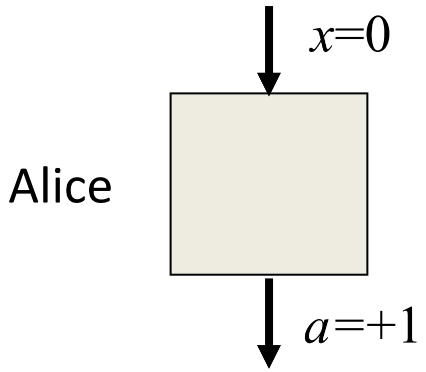
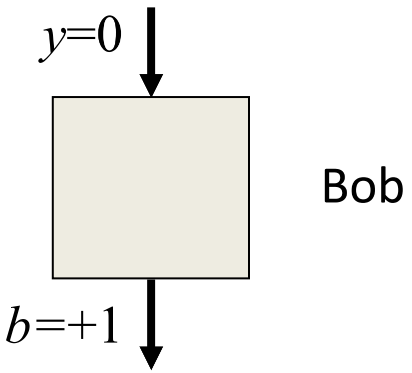


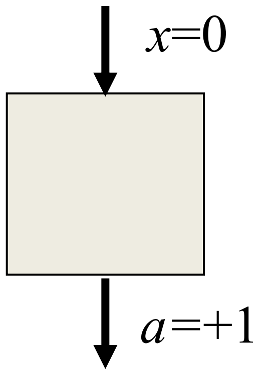
$$p(+1,+1|0,0) = \frac{1}{2}$$



$$p(ab|xy) = \begin{pmatrix} p(+1,+1|0,0) & p(+1,-1|0,0) & p(-1,+1|0,0) & p(-1,-1|0,0) \\ p(+1,+1|0,1) & p(+1,-1|0,1) & p(-1,+1|0,1) & p(-1,-1|0,1) \\ p(+1,+1|1,0) & p(+1,-1|1,0) & p(-1,+1|1,0) & p(-1,-1|1,0) \\ p(+1,+1|1,1) & p(+1,-1|1,1) & p(-1,+1|1,1) & p(-1,-1|1,1) \end{pmatrix}$$

Example

Alice



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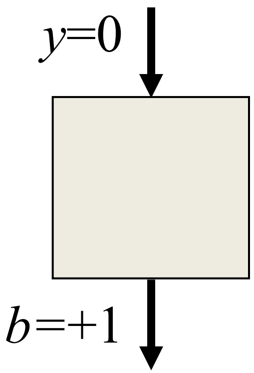
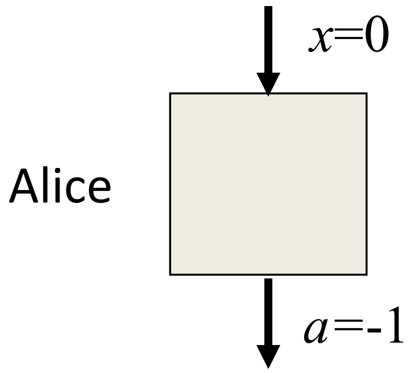


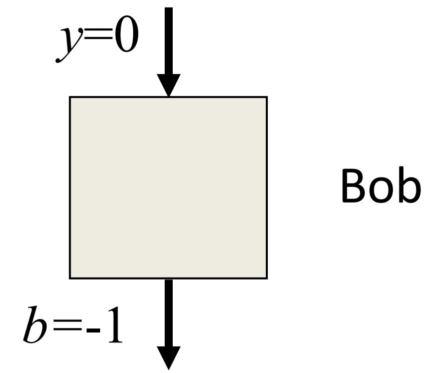
Bob

$$p(ab|xy) = \begin{pmatrix} 1/2 & p(+1,-1|0,0) & p(-1,+1|0,0) & p(-1,-1|0,0) \\ p(+1,+1|0,1) & p(+1,-1|0,1) & p(-1,+1|0,1) & p(-1,-1|0,1) \\ p(+1,+1|1,0) & p(+1,-1|1,0) & p(-1,+1|1,0) & p(-1,-1|1,0) \\ p(+1,+1|1,1) & p(+1,-1|1,1) & p(-1,+1|1,1) & p(-1,-1|1,1) \end{pmatrix}$$

Example



$$p(-1,-1|0,0) = \frac{1}{2}$$



$$p(ab|xy) = \begin{pmatrix} 1/2 & p(+1,-1|0,0) & p(-1,+1|0,0) & 1/2 \\ p(+1,+1|0,1) & p(+1,-1|0,1) & p(-1,+1|0,1) & p(-1,-1|0,1) \\ p(+1,+1|1,0) & p(+1,-1|1,0) & p(-1,+1|1,0) & p(-1,-1|1,0) \\ p(+1,+1|1,1) & p(+1,-1|1,1) & p(-1,+1|1,1) & p(-1,-1|1,1) \end{pmatrix}$$

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Example

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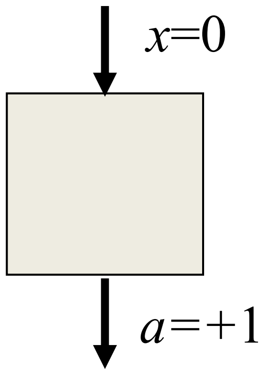
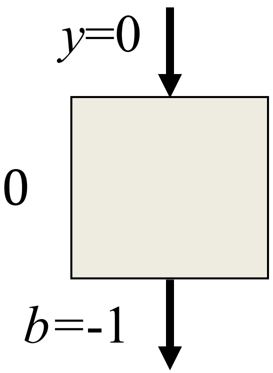


$$p(+1,-1|0,0) = p(-1,+1|0,0) = 0$$



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Physical correlations

Physical principles translate into limits on correlations.

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No-signalling correlations: correlations compatible with the no-signalling principle, i.e. the impossibility of instantaneous communication.

$$\sum_{a_{k+1},\ldots,a_N} p(a_1,\ldots,a_N | x_1,\ldots,x_N) = p(a_1,\ldots,a_k | x_1,\ldots,x_k)$$

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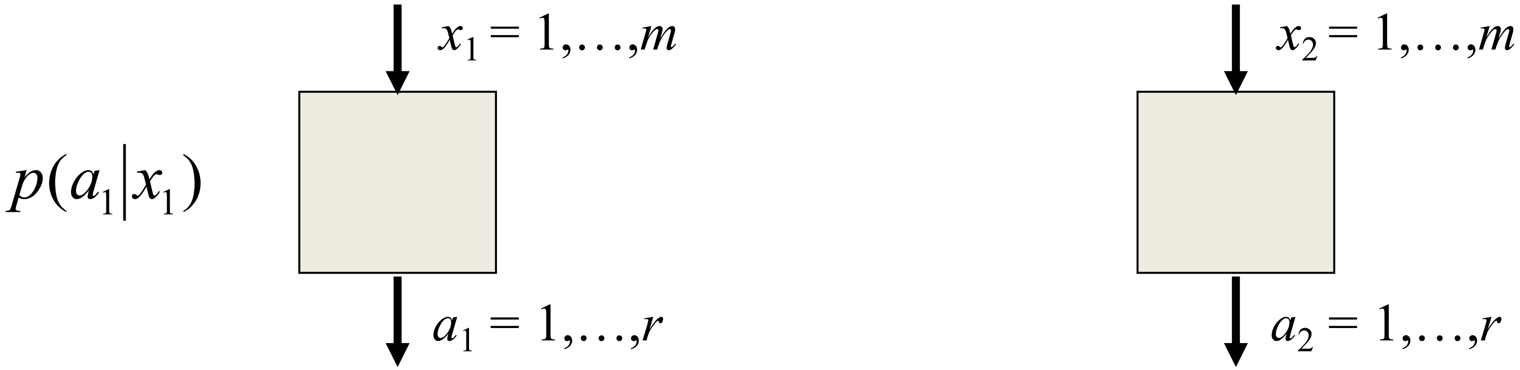

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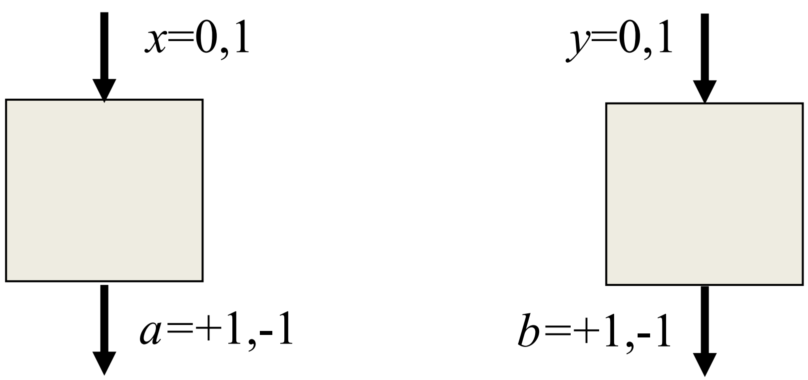

	++	+−	−+	−−
00	1/2	0	0	1/2
01	1/2	0	0	1/2
10	1/2	0	0	1/2
11	0	1/2	1/2	0

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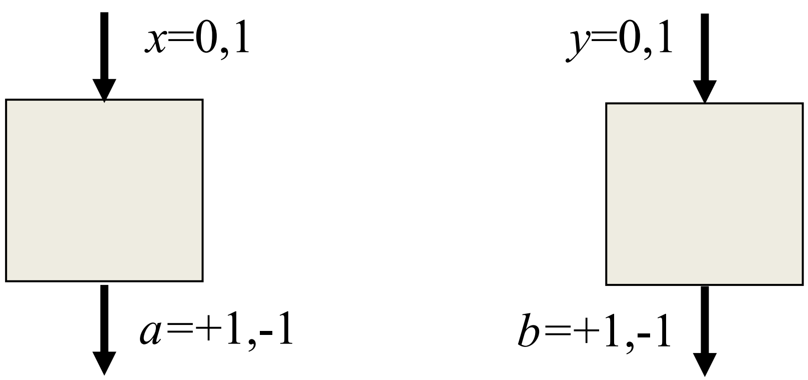

	+ +	+ −	− +	− −
00	1/2	0	0	1/2
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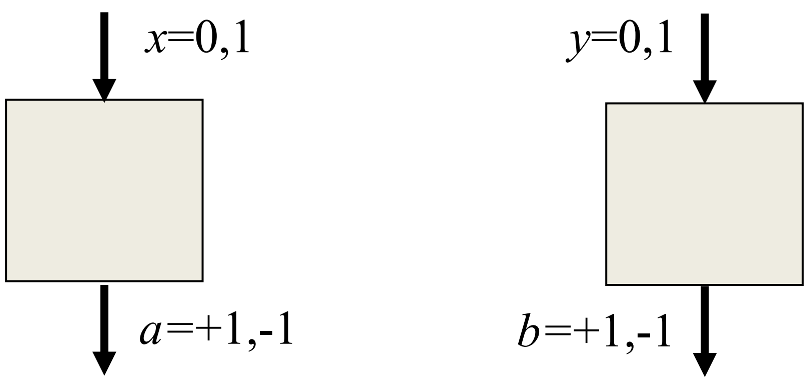

	$++$	$+-$	$-+$	$--$
00	1/2	0	0	1/2
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$$p_A(+1|0) = p(+1,+1|00) + p(+1,-1|00) = \frac{1}{2} = p(+1,+1|01) + p(+1,-1|01)$$

Physical correlations

Classical correlations: correlations established by classical means.

$$p\left(a_1,\ldots,a_N \middle| x_1,\ldots,x_N\right) = \sum_\lambda p(\lambda) D\left(a_1 \middle| x_1, \lambda\right)\ldots D\left(a_N \middle| x_N, \lambda\right)$$

These are the standard "EPR" correlations. Independently of fundamental issues, these are the correlations achievable by classical resources. Bell inequalities define the limits on these correlations.

Physical correlations

Quantum correlations: correlations established by quantum means.

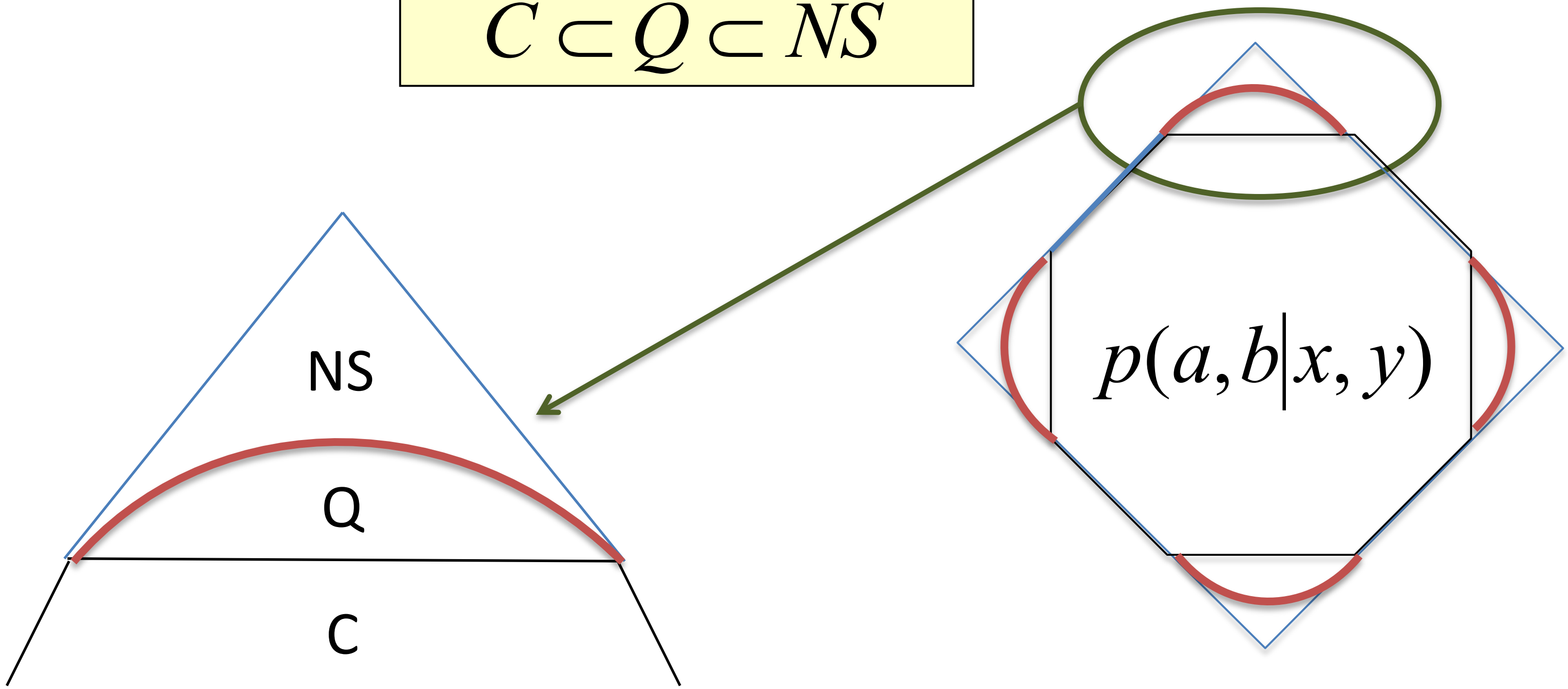
$$p(a_1,\ldots,a_N|x_1,\ldots,x_N) = \langle\Psi|M_{a_1}^{x_1}\otimes\cdots\otimes M_{a_N}^{x_N}|\Psi\rangle$$

$$\sum_{a_i} M_{a_i}^{x_i} = 1 \qquad M_{a'_i}^{x_i} M_{a_i}^{x_i} = \delta_{a_i a'_i} M_{a_i}^{x_i}$$

Everything is expressed in terms of operators (the quantum state and the measurement projectors) acting on a Hilbert space.

Physical correlations

$$C \subset Q \subset NS$$



Physical correlations

Bell

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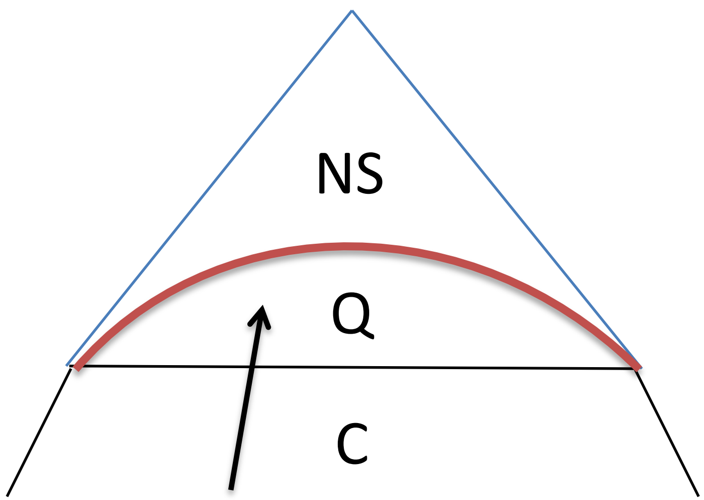
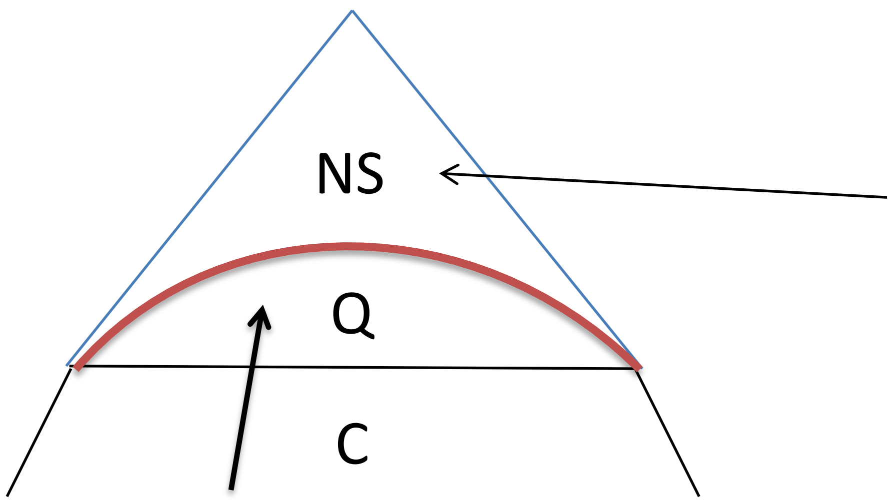


There exist correlations that cannot be explained by a classical model in which (deterministic) classical instructions specify the outcomes of the devices. These quantum correlations are known as **non-local** and they are detected by the violation of a Bell inequality.

Physical correlations

Bell $C \subset Q \subset NS$ Tsirelson

Popescu-Rohrlich



There exist correlations that are compatible with the no-signalling principle but cannot be obtained by performing local measurements on a quantum (entangled) state.

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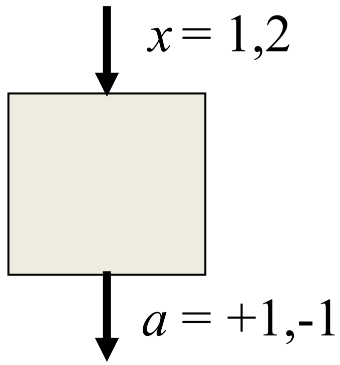
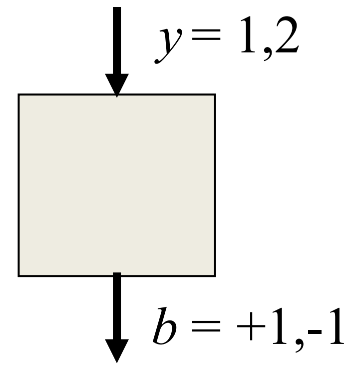
Example: CHSH scenario



$$CHSH = A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2$$



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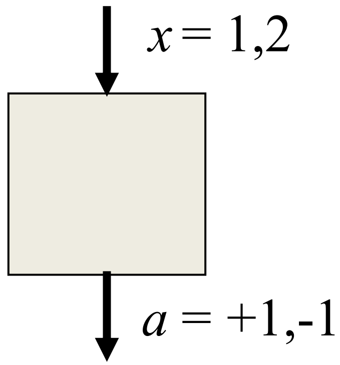

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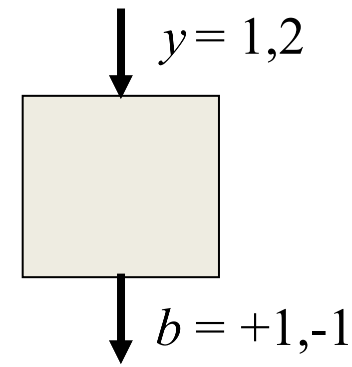
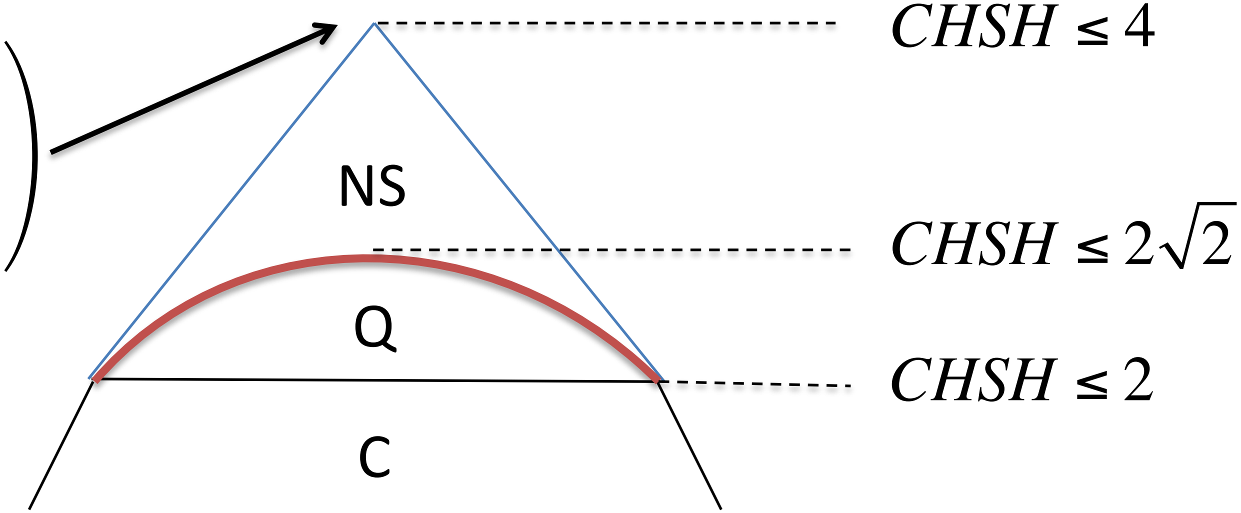


$$\begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$



Characterizing quantum correlations

Given $p(a,b|x,y)$, does it have a quantum realization?

$$p(a,b|x,y) = \langle \Psi | M_a^x \otimes M_b^y | \Psi \rangle$$

$$\sum_a M_a^x = 1$$

$$M_a^x M_{a'}^x = \delta_{a'a} M_a^x$$

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Example:

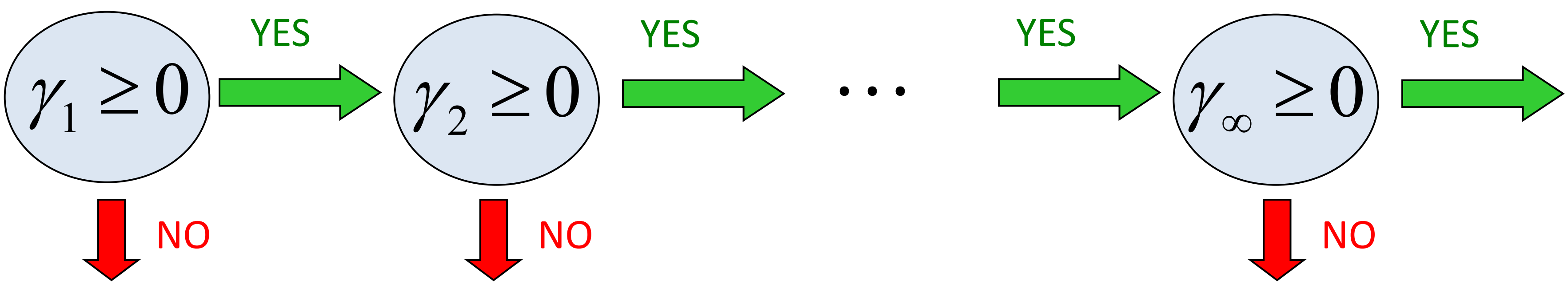
$$p\left(a,b\middle|0,0\right)=p\left(a,b\middle|0,1\right)=p\left(a,b\middle|1,0\right)=\frac{1}{8}\left(2+\sqrt{3},2-\sqrt{3},2-\sqrt{3},2+\sqrt{3}\right)$$

$$p\left(a,b\middle|1,1\right)=\left(0.245,0.255,0.255,0.245\right)$$

Previous work by Tsirelson

NPA hierarchy

Given a probability distribution $p(a,b|x,y)$, we have defined a hierarchy consisting of a series of tests based on semi-definite programming techniques allowing the detection of supra-quantum correlations.



The hierarchy is asymptotically convergent.

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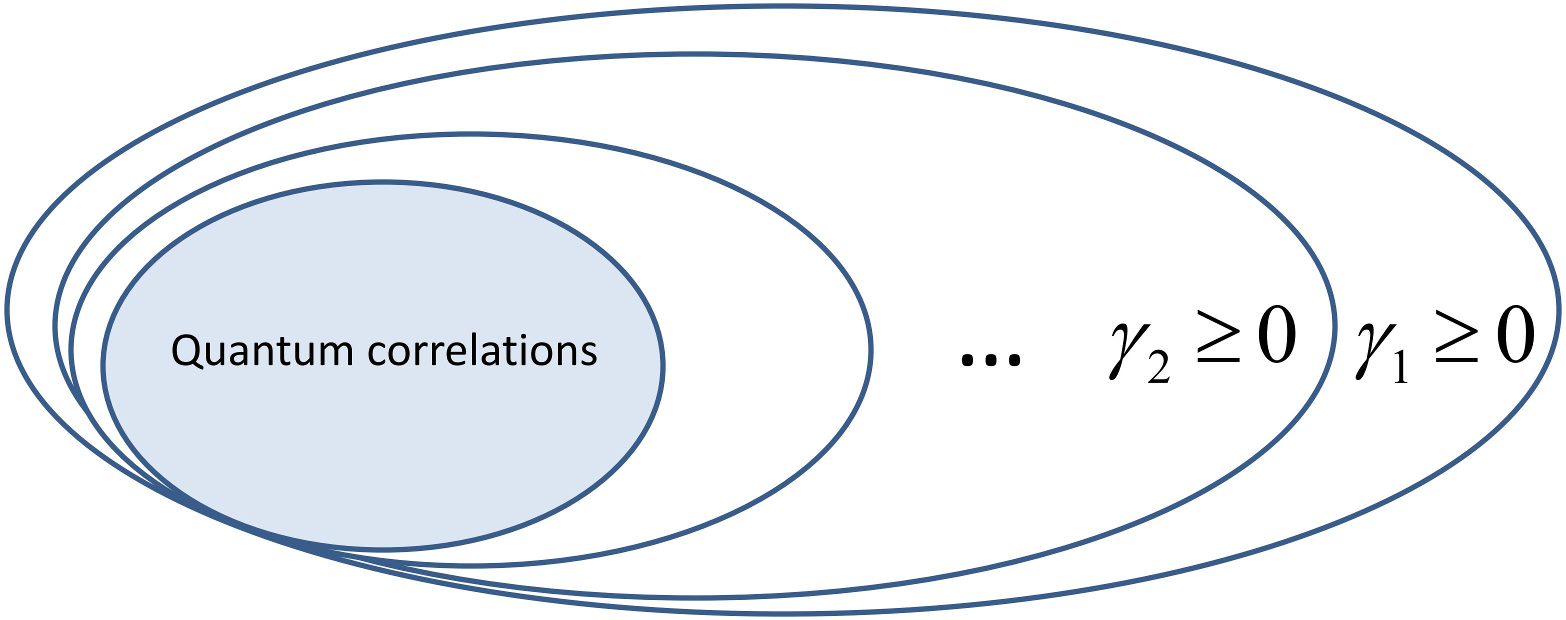
NPA hierarchy



Every step in the hierarchy defines a convex set that is included in the previous step. Convergence is provably attained asymptotically.

Characterizing quantum correlations

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$$p(a,b|1,1) = (0.245,0.255,0.255,0.245)$$

Solution: it is not quantum, that is, there exists no quantum state of two particles and local measurements acting on them that produce these correlations.

The experimental observation of these correlations would imply the failure of quantum physics, as Bell violations did for classical physics.

Going beyond NPA

Ground-state energies

A standard problem in physics is to find the ground state energy of a systems of N particles whose interactions are described by a Hamiltonian operator H.

$$\min_{|\psi\rangle} \langle\psi|H|\psi\rangle \qquad \text{subject to some constraints}$$

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If the problem can be cast as a non-commutative polynomial optimization, the previous hierarchy provides lower bounds. It complements the standard approach!

Classical spin systems

Classical spin problems: $$\min_{\vec{\sigma}} H(\vec{\sigma}) = \sum_{i,j} J_{i,j}\sigma_i\sigma_j + \sum_i h_i\sigma_i$$

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such that:

$\sigma_i^2 - 1 = 0$ for all i

Commutative polynomial optimization:

$$E_1 \leq E_2 \leq \cdots \leq E_\infty \to E_g \leq E_a$$

Quantum spin systems

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$$\sigma_i^\alpha - 1 = 0$$
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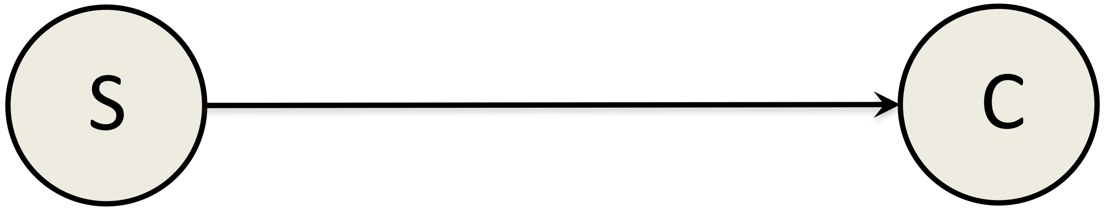
See Jie Wang's talk.

Causal networks



Main question: understand the causes that could be behind the observed correlations among a set of random variables.

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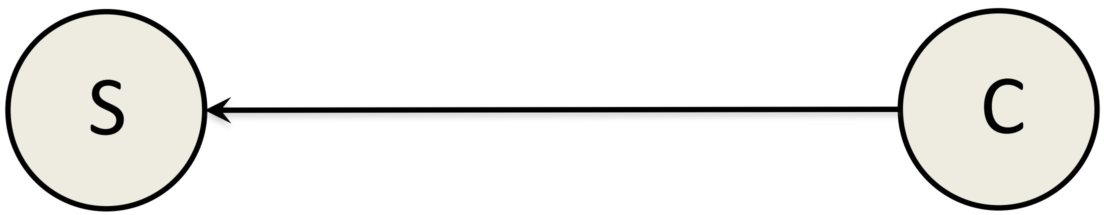

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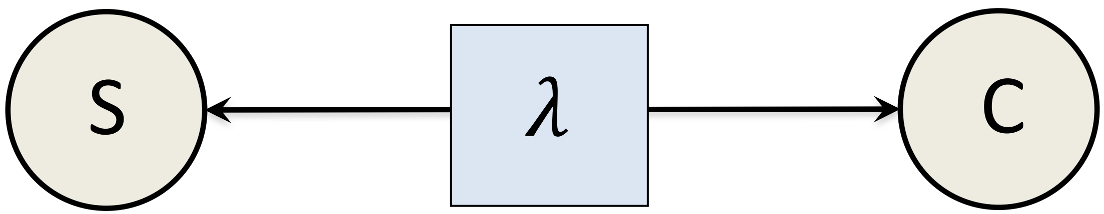

Main question: understand the causes that could be behind the observed correlations among a set of random variables.

Given two correlated variables, either direct causation is possible.

But even more intricate causation patterns could explain the correlations.

Causal networks

Representation of causality patterns through directed acyiclic graphs. Observed variables are represented by circles, hidden variables by squares and causes by directed edges.

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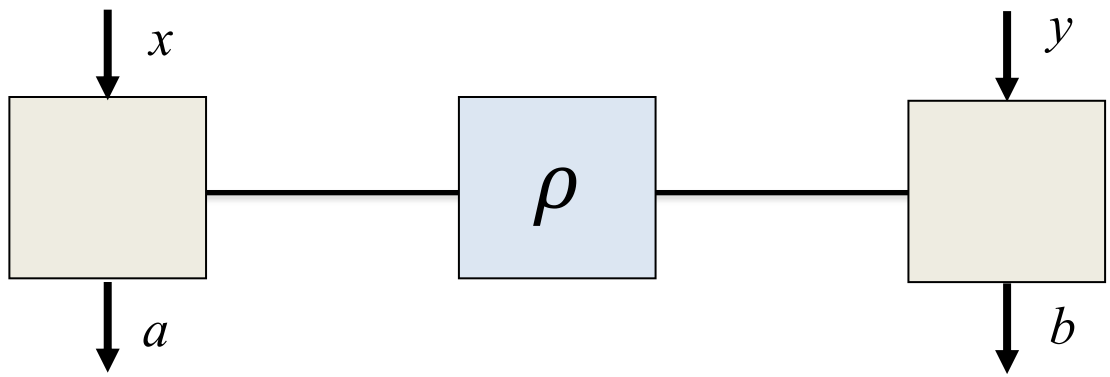
Bell setups can be understood in this language. Fritz, NJP'12; Wood & Spekkens, NJP '15



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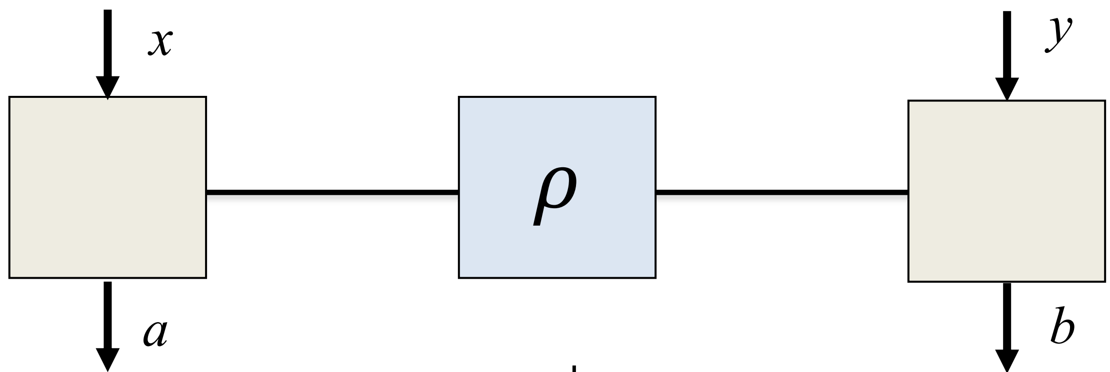
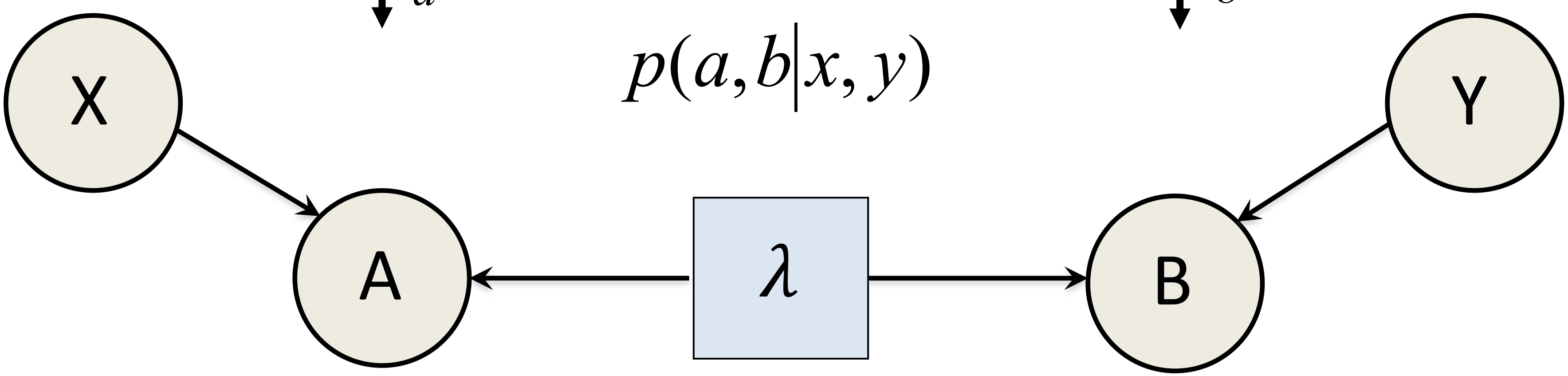


$$p(a,b|x,y)$$



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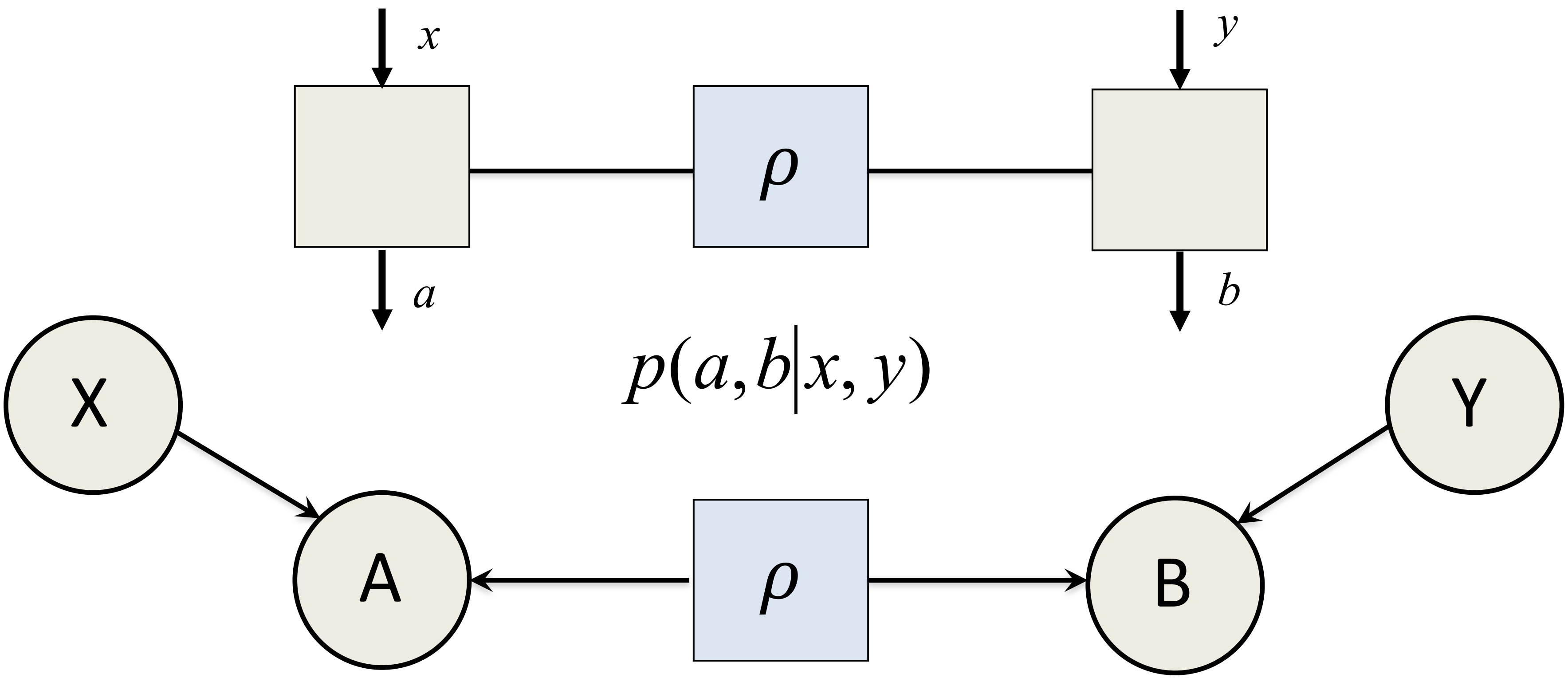

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Quantum causality

Bell's theorem: nonlocal correlations can be explained by a quantum causal model, but not by the classical counterpart.

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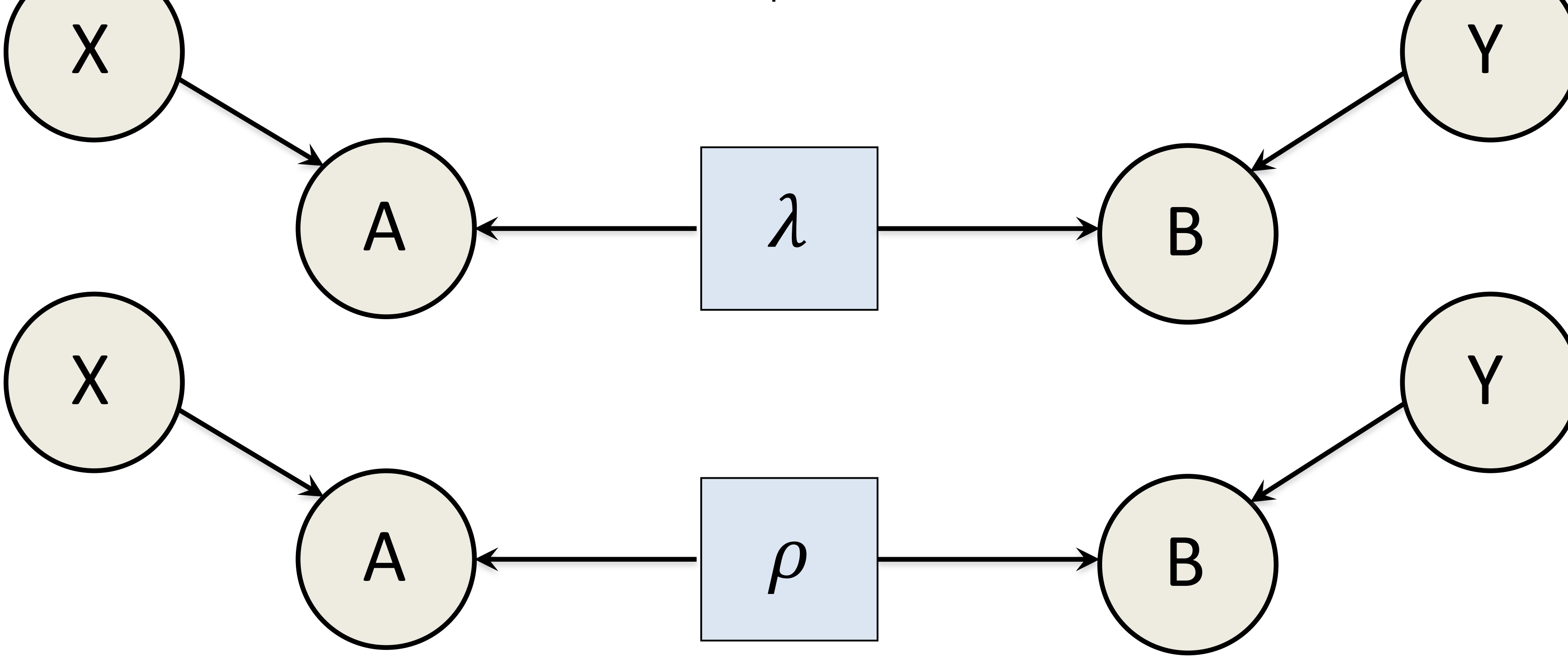

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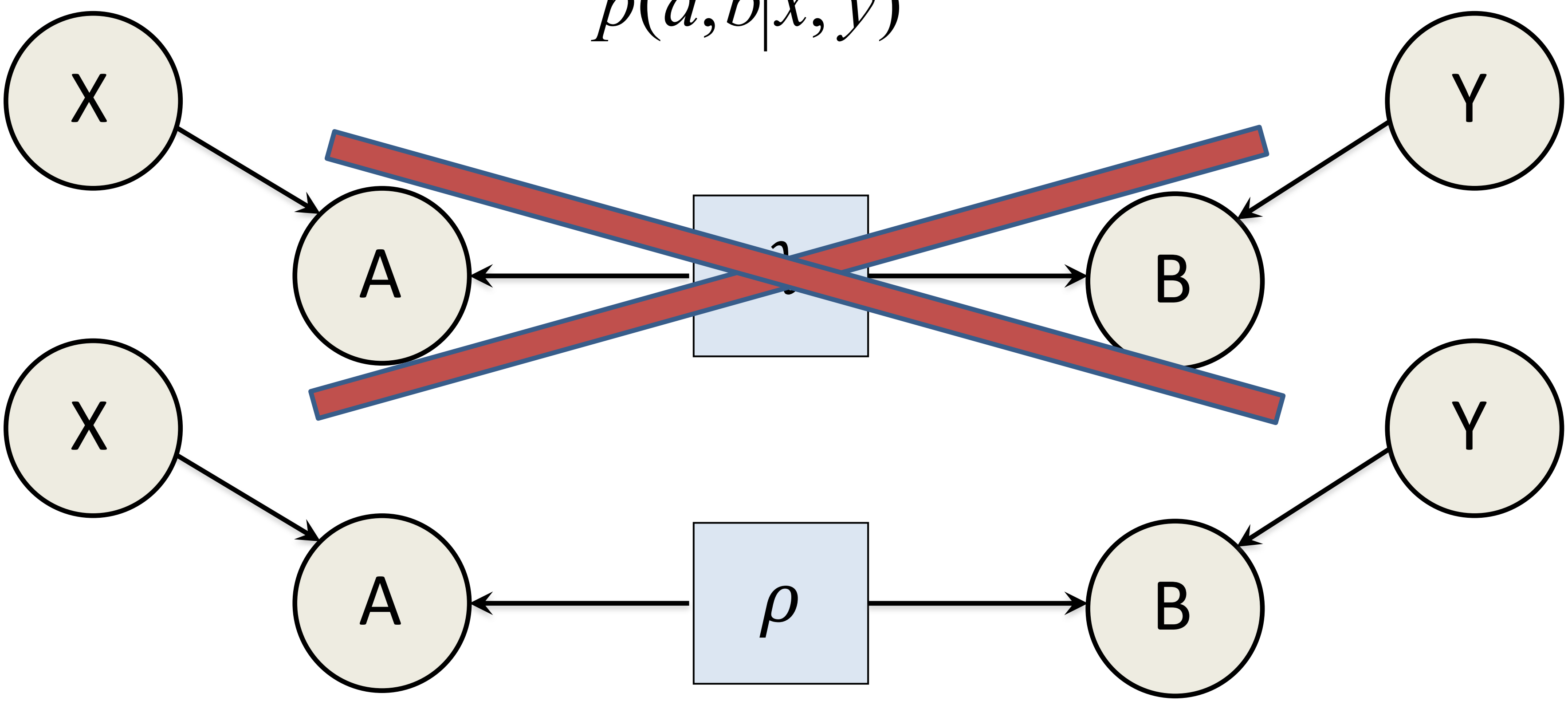
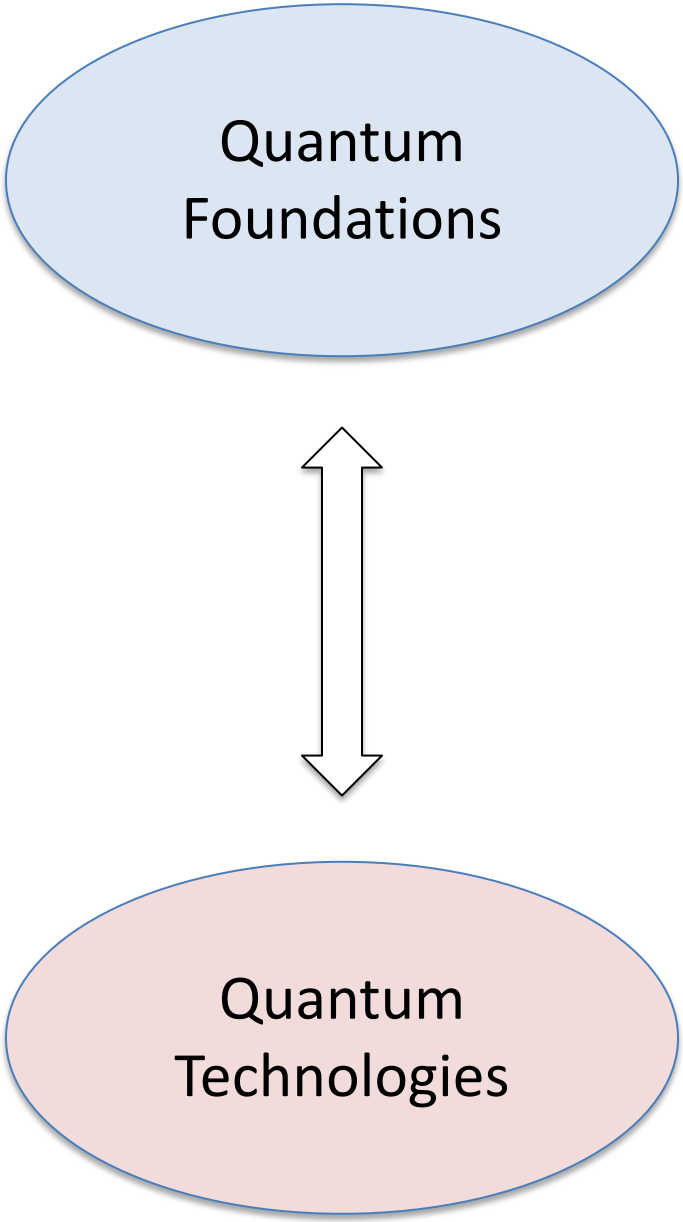


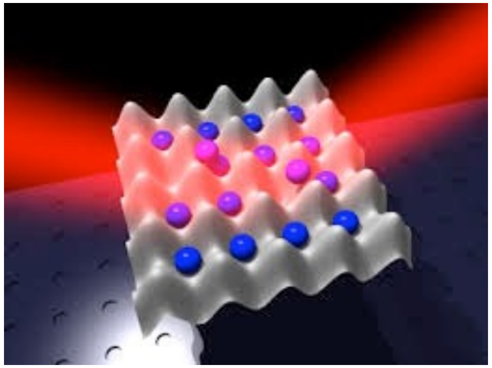
How is causality affected by quantum information?

See Alejandro Pozas-Kerstjens' talk.


Quantum Foundations
Quantum Technologies

Quantum information technologies

Quantum Computer



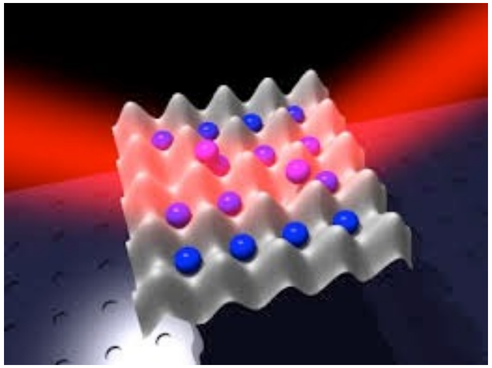
Quantum Simulator

Quantum Cryptography

QRNG

Quantum certification

Is this a Quantum Computer?



Does this properly simulate a quantum system?

Is this cryptographically secure?

Is this quantum random?