joint with L. Alon (IAS)

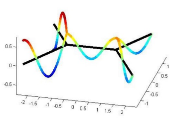
Metric (quantum) graphs - definitions

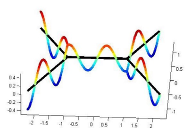
$\Gamma = (V, E)$ graph with $|E| < \infty$ $|V| < \infty$
Each edge corresponds to a finite interval $[0, l_e]$.

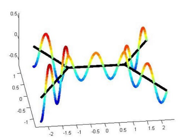
Operator is 1d Laplacian: $-\partial_e^2 f|_e = k^2 f|_e$ at each $e \in E$
with Neumann (Kirchhoff) vertex conditions:

- $\forall\, e_1, e_2 \sim v$ $\quad f|_{e_1}(v) = f|_{e_2}(v)$
- $\sum_{e \sim v} \partial_e f|_e(v) = 0$

The spectrum $\{k_n^2\}$ is discrete and $k_0 < k_1 \le k_2 \nearrow \infty$


k_8

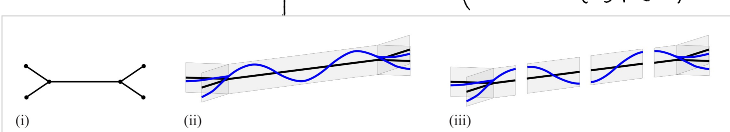

k_{13}


k_{16}

Neumann domains: definitions

Def:

- Neumann points of f are $\mathcal{N}(f) := \{ x \in \Gamma \setminus V : f'(x) = 0 \}$
- A Neumann domain, Ω, of f is the closure of a connected component of $\Gamma \setminus \mathcal{N}(f)$ [Ω is a graph!]



Assumption: edge lengths are rationaly indep. [i.e., linearly independent over $\mathbb{Q}$]

Assumption (generic)

Def: f_n is generic if

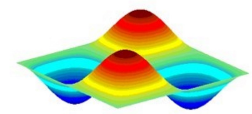
(1) λ_n is simple
(2) $f_n \neq 0$ at vertices
(3) $\partial f_n \neq 0$ at internal vertices

For fixed (rationally indep.) edge lengths, almost all eigenfunctions are generic.
For almost all edge lengths - all efuncs are generic
[Berkolaiko-Liu '16, Alon - in-prep]

1st aim - count Neumann points
bounds & statistics

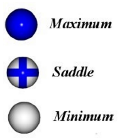
Neumann domains on manifolds

"Topography" of the function

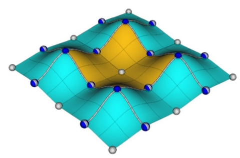


Nodal domains - regions above \ below sea level

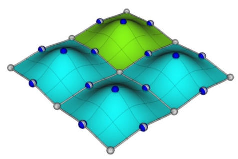
Neumann domains - where does water flow?



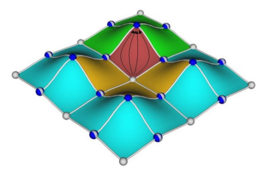
stable manifold (of minimum)



Unstable manifold (of maximum)



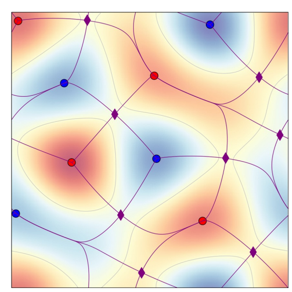
Their intersection



is a Neumann domain

Figures by Attila Gabor Gyulassy.

- Flow is along $-\nabla f$
- Neumann domain is intersection of stable manifold of minimum with unstable manifold of maximum
- Neumann lines are flow lines connected to saddle points



f is Laplacian eigenfunction on 2d flat torus

$f|_\Omega$ satisfies Neumann boundary conditions at $\partial\Omega$.
Is $f|_\Omega$ first efunction of Neumann Laplacian on Ω?

Questions

- count
- geometry
- any question from nodal

Joint works with Bersudsky, Cox, Egger, Fajman, Taylor

- Idea to consider Neumann domains (in spectral theory) in Zelditch '13; McDonald-Fulling '14

- Works on counting critical points (deterministic, not probabilistic) Jakobson-Nadirashvili '99; Buhovsky-Logunov-Sodin '19, Bérard-Charron-Helffer '20

Back to graphs (why interesting?)

- Nodal surplus $\sigma(n) := \begin{pmatrix} \text{nodal point} \\ \text{count of } f_n \end{pmatrix} - n$ $\quad (0 \le \sigma(n) \le \beta)$, $\beta = E - V + 1$
- Neumann surplus $\omega(n) := \begin{pmatrix} \text{Neumann point} \\ \text{count of } f_n \end{pmatrix} - n$

Thm [Alon, B. '21]
Let f_n be a generic eigenfunction.
Then

$$1-\beta-|\partial\Gamma| \le \omega(n) - \sigma(n) \le \beta - 1$$

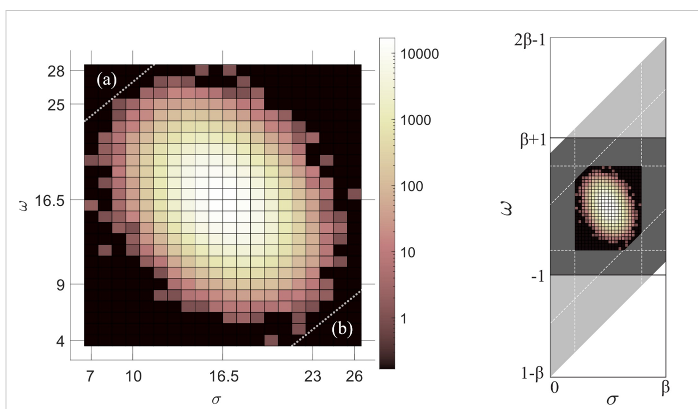
$|\partial\Gamma| :=$ # of degree one vertices of Γ

$$1-\beta-|\partial\Gamma| \le \omega(n) \le 2\beta - 1$$

Another proof (partition approach) in Hofmann-Kennedy
Partitions also in Kennedy-Kurasov-Léna-Mugnolo '20, B.-Berkolaiko-Raz-Smilansky '12, Hofmann-Kennedy-Mugnolo-Plümer 20

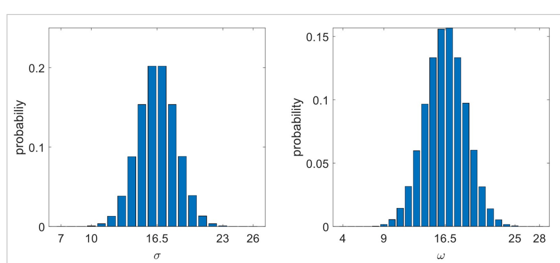
Conjecture: $-(|\partial\Gamma|+1) \le \omega(n) \le \beta + 1$ (stricter bounds if $\beta > 2$)

support: numerics, stowers, mandarins (also, on trees, bounds of thm are strict)



Bihistogram of σ & ω for Γ with $\beta = 33$, $|\partial\Gamma| = 0$ (6-regular with $|V| = 16$)

Probability distribution of $\omega(n)$



Def:

- $G := \{ n \in \mathbb{N} : f_n \text{ is generic} \}$

- $G := \{ n \in \mathbb{N} : f_n \text{ is generic} \}$
- For $A \subset G$, $\quad d_G(A) := \lim\limits_{N \to \infty} \dfrac{| A \cap \{1, 2, \dots, N\} |}{| G \cap \{1, 2, \dots, N\} |}$ density with respect to generic set

Thm

- $\omega : G \to \mathbb{Z}$ is a finite random variable with respect to d_G.
  Its probability distribution is $\quad \mathbb{P}(\omega = j) := d_G(\omega^{-1}(j))$.
- $\mathbb{P}$ is symmetric around $\frac{1}{2}(\beta - |\partial\Gamma|)$.

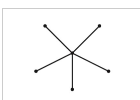
$$\mathbb{P}(\omega = j) = \mathbb{P}(\omega = \beta - |\partial\Gamma| - j)$$

Corollary

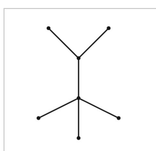
$$\mathbb{E}[\omega] = \tfrac{1}{2}(\beta - |\partial\Gamma|)$$

Similar to $\quad \mathbb{E}[\sigma] = \frac{1}{2}\beta \quad$ (Alon-B.-Berkolaiko)

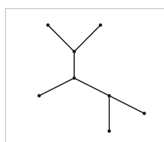
Implication for inverse problems (e.g. tree graphs)



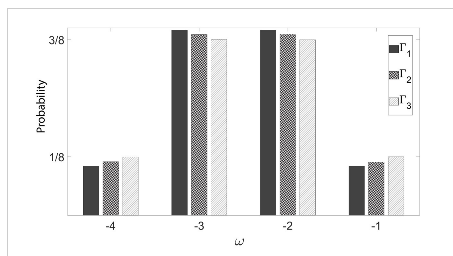
Γ_1



Γ_2



Γ_3



Probability dist' of ω for 3 trees with $|\partial\Gamma| = 5$
(calculated for $\sim 10^6$ eigenfunctions per graph)

* What do we know about probability distribution of ω ?

Thm

Let Γ be a tree graph whose vertices are of degrees 1 or 3.
The $-1-\omega$ is a binomial random variable:

$$\forall \quad 1 - |\partial\Gamma| \le j \le -1 \qquad \mathbb{P}(\omega = j) = \binom{|\partial\Gamma| - 2}{-j-1} \, 2^{2 - |\partial\Gamma|}$$

Conjecture (universality)

Let $\{\Gamma^{(m)}\}$ sequence of graphs with $\{\beta^{(m)}\}$ and $\{|\partial\Gamma^{(m)}|\}$
$\omega^{(m)}$ is the Neumann surplus r.v. of $\Gamma^{(m)}$.
If $\quad \beta^{(m)} + |\partial\Gamma^{(m)}| \longrightarrow \infty \quad$ then $\quad \dfrac{\omega^{(m)} - \mathbb{E}[\omega^{(m)}]}{\sqrt{\mathbb{V}[\omega^{(m)}]}} \xrightarrow[m \to \infty]{D} N(0,1)$

If $\beta^{(m)} + |\partial\Gamma^{(m)}| \longrightarrow \infty$, then $\dfrac{\omega^{(m)} - \mathbb{E}[\omega^{(m)}]}{\sqrt{\mathbb{V}[\omega^{(m)}]}} \xrightarrow[m\to\infty]{D} N(0,1)$

support: numerics & thm above & connection to nodal surplus

Methods and tools of proofs

* Characteristic torus & secular manifold - ergodic thm

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} h(n) = \int_{\Sigma} \tilde{h}(\kappa)\, d\mu_{\ell}$$

* Existence of oracle function on secular manifold

  - But for Neumann domains no index theorem (yet?) such as the nodal-magnetic connection [Berkolaiko '13; Colin-de-Verdière '13; Berkolaiko-Weyand '14]

* Connection to local observables:

Spectral position

Def: Let $v \in V$. $\forall n \in G$ $\exists!$ Neumann domain Ω which contains v.

$N^{(v)}(n) :=$ # nodal points of $f_n|_{\Omega}$

(It also equals the position of k_n in the spectrum of Ω)

$$\sum_{v \in V \setminus \partial\Gamma} N^{(v)} = \sigma - \omega + (|E| - |\partial\Gamma|)$$

$N^{(v)}$ is symmetrically supported on $[1, \dots, \deg(v)]$

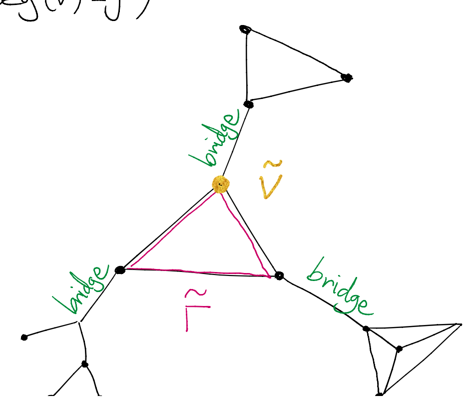
$$\mathbb{P}(N^{(v)} = j) = \mathbb{P}(N^{(v)} = \deg(v) - j)$$

Proof for "3-regular" trees

Prop $\tilde{\Gamma}$ subgraph of Γ

$\tilde{\Gamma}$ separated from $\Gamma \setminus \tilde{\Gamma}$ by bridges

$\vec{N}^{(\Gamma\setminus\tilde{\Gamma})} := \left(N^{(v)}\right)_{v \in \Gamma \setminus \tilde{\Gamma}}$



$$\vec{N}^{(\Gamma \setminus \tilde{\Gamma})} := \left(N^{(v)} \right)_{v \in \Gamma \setminus \tilde{\Gamma}}$$

Then $\mathbb{P}\left(N^{(\tilde{v})} = j \mid \vec{N}^{(\Gamma \setminus \tilde{\Gamma})}\right) = \mathbb{P}\left(N^{(\tilde{v})} = \deg(\tilde{v}) - j \mid \vec{N}^{(\Gamma \setminus \tilde{\Gamma})}\right)$

<u>Corollaries</u>

(1) Γ is tree graph. $u, v \in V \setminus \partial\Gamma$. $u \neq v$
Then $N^{(u)}, N^{(v)}$ uncorrelated random variables

(2) Γ is "3-regular" tree. All $\{N^{(v)}\}_{v \in V \setminus \partial\Gamma}$ are mutually independent.

(3) $-1-\omega = \sum_{v \in V \setminus \partial\Gamma} (N^{(v)} - 1)$ is binomial