Non-isothermal viscoelastic flows with conservation laws and relaxation

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Hyperbolic systems of balance laws

System of balance laws

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{j=1}^{d} \frac{\partial \boldsymbol{f}_j(\boldsymbol{u})}{\partial x_j} = \boldsymbol{c}(\boldsymbol{u})$$

$\boldsymbol{u} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^n$, $\boldsymbol{f}_j : \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{c} : \mathbb{R}^n \to \mathbb{R}^n$

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Quasilinear system

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{j=1}^{d} \boldsymbol{A}_j(\boldsymbol{u}) \frac{\partial \boldsymbol{u}}{\partial x_j} = \boldsymbol{c}(\boldsymbol{u}), \qquad \boldsymbol{A}_j(\boldsymbol{u}) \stackrel{\text{def}}{=} \frac{\partial \boldsymbol{f}_j(\boldsymbol{u})}{\partial \boldsymbol{u}}$$

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Definition (Symmetrizability)
Let $\mathcal{U}$ be an open subset of $\mathbb{R}^n$. The quasilinear system is called *symmetrizable* in $\mathcal{U}$ if there exists a C^∞ mapping $\boldsymbol{S} : \mathcal{U} \to \mathbb{R}^{n \times n}_{\text{sym}}$ such that for all $u \in \mathcal{U}$

- $\boldsymbol{S}(u)$ is positive definite,
- $\boldsymbol{S}(u)\boldsymbol{A}_j(u)$ are symmetric.

Short-time existence of smooth solutions [Benzoni-Gavage and Serre, 2006]

Theorem

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^n$. We assume that $\boldsymbol{A}^j$ and $\boldsymbol{c}$ are C^∞ functions of $\boldsymbol{u} \in \mathcal{U}$ and that the quasilinear system is symmetrizable in $\mathcal{U}$. Let $\boldsymbol{u}_0 \in \mathcal{U}$ and $\widetilde{\boldsymbol{u}_0} \in H^s(\mathbb{R}^d; \mathbb{R}^n)$ with $s > 1 + \frac{d}{2}$ such that $\boldsymbol{u}_0 + \widetilde{\boldsymbol{u}_0}$ is compactly supported in $\mathcal{U}$.

Then, there exists $T > 0$ and a unique classical solution $\boldsymbol{u} \in C^1(\mathbb{R}^d \times [0, T]; \mathcal{U})$ of the Cauchy problem associated with the quasilinear system and initial data $\boldsymbol{u}(0) = \boldsymbol{u}_0 + \widetilde{\boldsymbol{u}_0}$. Furthermore, $\boldsymbol{u} - \boldsymbol{u}_0 \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$.

Definition

Let $\mathcal{U}$ be a convex subset of $\mathbb{R}^d$. Then, a convex function $s : \mathcal{U} \to \mathbb{R}$ is called a mathematical entropy for the system of balance laws if there exist d functions $F_j : \mathcal{U} \to \mathbb{R}$, $1 \leq j \leq d$, called entropy fluxes, such that for all $\boldsymbol{u} \in \mathcal{U}$

$$\frac{\partial s(\boldsymbol{u})}{\partial \boldsymbol{u}} \frac{\partial \boldsymbol{f}_j(\boldsymbol{u})}{\partial \boldsymbol{u}} = \frac{\partial F_j(\boldsymbol{u})}{\partial \boldsymbol{u}}.$$

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Additional balance law

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{j=1}^{d} \frac{\partial \boldsymbol{f}_j(\boldsymbol{u})}{\partial x_j} = \boldsymbol{c}(\boldsymbol{u}) \implies \boxed{\frac{\partial s(\boldsymbol{u})}{\partial t} + \sum_{j=1}^{d} \frac{\partial F_j(\boldsymbol{u})}{\partial x_j} = b(\boldsymbol{u})} \implies \frac{\partial s(\boldsymbol{u})}{\partial \boldsymbol{u}} \frac{\partial \boldsymbol{f}_j(\boldsymbol{u})}{\partial \boldsymbol{u}} = \frac{\partial F_j(\boldsymbol{u})}{\partial \boldsymbol{u}}$$

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mathematical entropy $\sim$ “physical” entropy/energy

Euler equations (compressible, inviscid, non-heat-conducting fluid)

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$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) &= 0 \\ \frac{\partial (\rho \boldsymbol{v})}{\partial t} + \operatorname{div}(\rho \boldsymbol{v} \otimes \boldsymbol{v} + p \boldsymbol{I}) &= \rho \boldsymbol{f} \\ \frac{\partial (\rho \eta)}{\partial t} + \operatorname{div}(\rho \eta \boldsymbol{v}) &= 0 \end{aligned}$$

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- Complete equation of state

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- Mathematical entropy

$$s(\boldsymbol{u}) = \rho e_{\mathrm{tot}} = \rho \left(\frac{1}{2} |\boldsymbol{v}|^2 + e(\rho, \eta) \right)$$

Compressible heat-conducting Maxwell fluid

Balance laws of continuum thermomechanics

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\frac{\partial (\rho \eta)}{\partial t} + \operatorname{div}(\rho \eta \boldsymbol{v} + \boldsymbol{j}_\eta) &= \xi
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specific total energy … $e_{\mathrm{tot}} = \frac{1}{2}|\boldsymbol{v}|^2 + e$
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Constitutive relations and equation of state

Cauchy stress tensor	$\boldsymbol{T}$
energy flux	$\boldsymbol{j}_e$
entropy flux	$\boldsymbol{j}_\eta$
equation of state	$e = e(\rho, \eta, \dots)$

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Maxwell–Cattaneo law [Cattaneo, 2011], [Chandrasekharaiah, 1986], [Jou et al., 1999]

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- $\frac{\mathrm{d}}{\mathrm{d}t} \stackrel{\text{def}}{=} \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla$ denotes the material derivative

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- balance form of Maxwell–Cattaneo law

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Helmholtz free energy [Dressler et al., 1999]

$$\psi(\rho, \theta, \boldsymbol{j}_e, \boldsymbol{C}) \stackrel{\text{def}}{=} \psi_{\mathrm{s}}(\rho, \theta) + \frac{\tau_0}{2\kappa}|\boldsymbol{j}_e|^2 + \frac{\alpha}{2}(K \operatorname{tr} \boldsymbol{C} - k_{\mathrm{B}}\theta \log \det \boldsymbol{C})$$

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- Noble–Abel stiffened-gas fluid [Le Métayer and Saurel, 2016]

$$e_{\mathrm{s}}(\rho, \eta) = c_{\mathrm{V,s}}\theta_{\mathrm{ref}}\left(\frac{\rho}{\rho_{\mathrm{ref}}} \frac{1}{1 - b\rho}\right)^{\gamma-1} \mathrm{e}^{\frac{\eta}{c_{\mathrm{V,s}}}} + \left(\frac{1}{\rho} - b\right) p_\infty + q$$

Governing equations for $(\rho, \rho\eta, \rho\boldsymbol{j}_e, \rho\boldsymbol{v}, \rho\boldsymbol{C})$

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$$\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t} - (\nabla\boldsymbol{v})\boldsymbol{C} - \boldsymbol{C}(\nabla\boldsymbol{v})^\top = -\frac{4K}{\zeta}\boldsymbol{C} + \frac{4k_{\mathrm{B}}\theta}{\zeta}\boldsymbol{I}$$

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pressure $\qquad p(\rho, \eta, \boldsymbol{j}_e, \boldsymbol{C}) = \rho^2 \dfrac{\partial e}{\partial \rho}(\rho, \eta, \boldsymbol{j}_e, \boldsymbol{C})$

Cauchy stress tensor $\qquad \boldsymbol{T}(\rho, \eta, \boldsymbol{j}_e, \boldsymbol{C}) = -p\boldsymbol{I} + \alpha\rho(K\boldsymbol{C} - k_{\mathrm{B}}\theta\boldsymbol{I})$

entropy production $\qquad \xi(\rho, \eta, \boldsymbol{j}_e, \boldsymbol{C}) = \dfrac{|\boldsymbol{j}_e|^2}{\kappa\theta^2} + \dfrac{2\alpha\rho}{\zeta\theta}\left|K\boldsymbol{C}^{\frac{1}{2}} - k_{\mathrm{B}}\theta\boldsymbol{C}^{-\frac{1}{2}}\right|^2 \geq 0$

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New tensorial quantity [Kaye, 1962], [Bernstein et al., 1963], [Bernstein et al., 1964], [Boyaval, 2020]

$$\boldsymbol{A} \stackrel{\text{def}}{=} \boldsymbol{F}^{-1} \boldsymbol{C} \boldsymbol{F}^{-\top}$$

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Consequence of

- e.e. for conformation tensor: $\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t} - (\nabla \boldsymbol{v}) \boldsymbol{C} - \boldsymbol{C} (\nabla \boldsymbol{v})^{\top} = -\frac{4K}{\zeta} \boldsymbol{C} + \frac{4k_{\mathrm{B}}\theta}{\zeta} \boldsymbol{I}$

New tensorial quantity [Kaye, 1962], [Bernstein et al., 1963], [Bernstein et al., 1964], [Boyaval, 2020]

$$\boldsymbol{A} \stackrel{\text{def}}{=} \boldsymbol{F}^{-1} \boldsymbol{C} \boldsymbol{F}^{-\top}$$

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- e.e. for deformation gradient: $\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t} = (\nabla\boldsymbol{v})\boldsymbol{F}$

Evolution equation for $\boldsymbol{F}$

$$\boxed{\frac{\partial(\rho\boldsymbol{F})}{\partial t} + \operatorname{div}\Big(\rho\Big(\boldsymbol{F}\otimes\boldsymbol{v} - \boldsymbol{v}\otimes\boldsymbol{F}^{\top}\Big)\Big) = \boldsymbol{0}}$$

New tensorial quantity [Kaye, 1962], [Bernstein et al., 1963], [Bernstein et al., 1964], [Boyaval, 2020]

$$\boldsymbol{A} \stackrel{\text{def}}{=} \boldsymbol{F}^{-1} \boldsymbol{C} \boldsymbol{F}^{-\top}$$

Evolution equation for $\boldsymbol{A}$

$$\boxed{\frac{\partial(\rho \boldsymbol{A})}{\partial t} + \operatorname{div}(\rho \boldsymbol{A} \otimes \boldsymbol{v}) = -\frac{4\rho K}{\zeta} \boldsymbol{A} + \frac{4\rho k_{\mathrm{B}} \theta}{\zeta} \boldsymbol{F}^{-1} \boldsymbol{F}^{-\top}}$$

Consequence of

- e.e. for conformation tensor: $\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t} - (\nabla \boldsymbol{v}) \boldsymbol{C} - \boldsymbol{C}(\nabla \boldsymbol{v})^{\top} = -\frac{4K}{\zeta} \boldsymbol{C} + \frac{4k_{\mathrm{B}}\theta}{\zeta} \boldsymbol{I}$
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Evolution equation for $\boldsymbol{F}$

$$\boxed{\frac{\partial(\rho \boldsymbol{F})}{\partial t} + \operatorname{div}\left(\rho\left(\boldsymbol{F} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{F}^{\top}\right)\right) = \boldsymbol{0}}$$

Consequence of

- Eulerian version of Piola’s identity: $\operatorname{div}\left(\frac{\boldsymbol{F}^{\top}}{\det \boldsymbol{F}}\right) = 0$

New tensorial quantity [Kaye, 1962], [Bernstein et al., 1963], [Bernstein et al., 1964], [Boyaval, 2020]

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Evolution equation for $\boldsymbol{A}$

$$\frac{\partial(\rho\boldsymbol{A})}{\partial t} + \operatorname{div}(\rho\boldsymbol{A}\otimes\boldsymbol{v}) = -\frac{4\rho K}{\zeta}\boldsymbol{A} + \frac{4\rho k_{\mathrm{B}}\theta}{\zeta}\boldsymbol{F}^{-1}\boldsymbol{F}^{-\top}$$

Consequence of

- e.e. for conformation tensor: $\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t} - (\nabla\boldsymbol{v})\boldsymbol{C} - \boldsymbol{C}(\nabla\boldsymbol{v})^{\top} = -\frac{4K}{\zeta}\boldsymbol{C} + \frac{4k_{\mathrm{B}}\theta}{\zeta}\boldsymbol{I}$
- e.e. for deformation gradient: $\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t} = (\nabla\boldsymbol{v})\boldsymbol{F}$

Evolution equation for $\boldsymbol{F}$

$$\frac{\partial(\rho\boldsymbol{F})}{\partial t} + \operatorname{div}\left(\rho\left(\boldsymbol{F}\otimes\boldsymbol{v} - \boldsymbol{v}\otimes\boldsymbol{F}^{\top}\right)\right) = \boldsymbol{0}$$

Consequence of

- Eulerian version of Piola's identity: $\operatorname{div}\left(\frac{\boldsymbol{F}^{\top}}{\det\boldsymbol{F}}\right) = 0$
- homogeneous material: $\rho_{\mathrm{R}} \stackrel{\text{def}}{=} \rho\det\boldsymbol{F} = \text{const}$

Governing equations for $(\rho, \rho\eta, \rho\boldsymbol{j}_e, \rho\boldsymbol{v}, \rho\boldsymbol{F}, \rho\boldsymbol{A})$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0$$

$$\frac{\partial (\rho\eta)}{\partial t} + \operatorname{div}\left(\rho\eta\boldsymbol{v} + \frac{\boldsymbol{j}_e}{\theta}\right) = \xi$$

$$\frac{\partial (\rho\boldsymbol{j}_e)}{\partial t} + \operatorname{div}\left(\rho\boldsymbol{j}_e \otimes \boldsymbol{v} + \frac{\kappa}{\tau_0}(\log\theta)\boldsymbol{I}\right) = -\frac{\boldsymbol{j}_e}{\tau_0\theta}$$

$$\frac{\partial (\rho\boldsymbol{v})}{\partial t} + \operatorname{div}(\rho\boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{T}) = \rho\boldsymbol{f}$$

$$\frac{\partial (\rho\boldsymbol{F})}{\partial t} + \operatorname{div}\left(\rho\left(\boldsymbol{F} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{F}^\top\right)\right) = \boldsymbol{0}$$

$$\frac{\partial (\rho\boldsymbol{A})}{\partial t} + \operatorname{div}(\rho\boldsymbol{A} \otimes \boldsymbol{v}) = -\frac{4\rho K}{\zeta}\boldsymbol{A} + \frac{4\rho k_{\mathrm{B}}\theta}{\zeta}\boldsymbol{F}^{-1}\boldsymbol{F}^{-\top}$$

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$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\boldsymbol{v}) = 0$$

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$$\frac{\partial(\rho\boldsymbol{v})}{\partial t} + \operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v} - \boldsymbol{T}) = \rho\boldsymbol{f}$$

$$\frac{\partial(\rho\boldsymbol{F})}{\partial t} + \operatorname{div}\left(\rho\left(\boldsymbol{F}\otimes\boldsymbol{v} - \boldsymbol{v}\otimes\boldsymbol{F}^\top\right)\right) = \boldsymbol{0}$$

$$\frac{\partial(\rho\boldsymbol{A})}{\partial t} + \operatorname{div}(\rho\boldsymbol{A}\otimes\boldsymbol{v}) = -\frac{4\rho K}{\zeta}\boldsymbol{A} + \frac{4\rho k_{\mathrm{B}}\theta}{\zeta}\boldsymbol{F}^{-1}\boldsymbol{F}^{-\top}$$

Balance of total energy

$$\frac{\partial(\rho e_{\mathrm{tot}})}{\partial t} + \operatorname{div}((\rho e_{\mathrm{tot}}\boldsymbol{I} - \boldsymbol{T})\boldsymbol{v} + \boldsymbol{j}_e) = \rho\boldsymbol{f}\cdot\boldsymbol{v}$$

Governing equations for $(\rho, \rho\eta, \rho\boldsymbol{j}_e, \rho\boldsymbol{v}, \rho\boldsymbol{F}, \rho\boldsymbol{A})$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0$$

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$$\frac{\partial (\rho\boldsymbol{v})}{\partial t} + \operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v} - \boldsymbol{T}) = \rho\boldsymbol{f}$$

$$\frac{\partial (\rho\boldsymbol{F})}{\partial t} + \operatorname{div}\left(\rho\left(\boldsymbol{F}\otimes\boldsymbol{v} - \boldsymbol{v}\otimes\boldsymbol{F}^\top\right)\right) = \boldsymbol{0}$$

$$\frac{\partial (\rho\boldsymbol{A})}{\partial t} + \operatorname{div}(\rho\boldsymbol{A}\otimes\boldsymbol{v}) = -\frac{4\rho K}{\zeta}\boldsymbol{A} + \frac{4\rho k_{\mathrm{B}}\theta}{\zeta}\boldsymbol{F}^{-1}\boldsymbol{F}^{-\top}$$

Balance of total energy

$$\frac{\partial (\rho e_{\mathrm{tot}})}{\partial t} + \operatorname{div}((\rho e_{\mathrm{tot}}\boldsymbol{I} - \boldsymbol{T})\boldsymbol{v} + \boldsymbol{j}_e) = \rho\boldsymbol{f}\cdot\boldsymbol{v}$$

Mathematical entropy

$$\rho e_{\mathrm{tot}} = \rho\left[\frac{1}{2}|\boldsymbol{v}|^2 + e_{\mathrm{s}}\left(\rho, \eta - \frac{\alpha}{2}k_{\mathrm{B}}\log\det(\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^\top)\right) + \frac{\tau_0}{2\kappa}|\boldsymbol{j}_e|^2 + \frac{\alpha}{2}K\operatorname{tr}(\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^\top)\right]$$

Governing equations for $(\rho, \rho\eta, \rho\boldsymbol{j}_e, \rho\boldsymbol{v}, \rho\boldsymbol{F}, \rho\boldsymbol{A})$

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\boldsymbol{v}) = 0$$

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$$\frac{\partial(\rho\boldsymbol{j}_e)}{\partial t} + \operatorname{div}\left(\rho\boldsymbol{j}_e \otimes \boldsymbol{v} + \frac{\kappa}{\tau_0}(\log\theta)\boldsymbol{I}\right) = -\frac{\boldsymbol{j}_e}{\tau_0\theta}$$

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$$\frac{\partial(\rho\boldsymbol{F})}{\partial t} + \operatorname{div}\left(\rho\left(\boldsymbol{F}\otimes\boldsymbol{v} - \boldsymbol{v}\otimes\boldsymbol{F}^\top\right)\right) = \boldsymbol{0}$$

$$\frac{\partial(\rho\boldsymbol{A})}{\partial t} + \operatorname{div}(\rho\boldsymbol{A}\otimes\boldsymbol{v}) = -\frac{4\rho K}{\zeta}\boldsymbol{A} + \frac{4\rho k_{\mathrm{B}}\theta}{\zeta}\boldsymbol{F}^{-1}\boldsymbol{F}^{-\top}$$

Balance of total energy

$$\frac{\partial(\rho e_{\mathrm{tot}})}{\partial t} + \operatorname{div}((\rho e_{\mathrm{tot}}\boldsymbol{I} - \boldsymbol{T})\boldsymbol{v} + \boldsymbol{j}_e) = \rho\boldsymbol{f}\cdot\boldsymbol{v}$$

Mathematical entropy

$$\rho e_{\mathrm{tot}} = \rho\left[\frac{1}{2}|\boldsymbol{v}|^2 + e_{\mathrm{s}}\left(\rho, \eta - \frac{\alpha}{2}k_{\mathrm{B}}\log\det(\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^\top)\right) + \frac{\tau_0}{2\kappa}|\boldsymbol{j}_e|^2 + \frac{\alpha}{2}K\operatorname{tr}(\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^\top)\right]$$

Yet another tensorial quantity

$$\boldsymbol{Y} \stackrel{\text{def}}{=} \boldsymbol{A}^{-2}, \qquad Y \stackrel{\text{def}}{=} \det(\boldsymbol{Y}^{-1})$$

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$$\boldsymbol{Y} \stackrel{\text{def}}{=} \boldsymbol{A}^{-2}, \qquad Y \stackrel{\text{def}}{=} \det(\boldsymbol{Y}^{-1})$$

Governing equations for $(\rho, \rho\eta, \rho\boldsymbol{j}_e, \rho\boldsymbol{v}, \rho\boldsymbol{F}, \rho\boldsymbol{Y}, \rho Y)$

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\boldsymbol{v}) &= 0 \\
\frac{\partial(\rho\eta)}{\partial t} + \operatorname{div}\left(\rho\eta\boldsymbol{v} + \frac{\boldsymbol{j}_e}{\theta}\right) &= \xi \\
\frac{\partial(\rho\boldsymbol{j}_e)}{\partial t} + \operatorname{div}\left(\rho\boldsymbol{j}_e \otimes \boldsymbol{v} + \frac{\kappa}{\tau_0}(\log\theta)\boldsymbol{I}\right) &= -\frac{\boldsymbol{j}_e}{\tau_0\theta} \\
\frac{\partial(\rho\boldsymbol{v})}{\partial t} + \operatorname{div}(\rho\boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{T}) &= \rho\boldsymbol{f} \\
\frac{\partial(\rho\boldsymbol{F})}{\partial t} + \operatorname{div}\left(\rho\left(\boldsymbol{F} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{F}^\top\right)\right) &= \boldsymbol{0} \\
\frac{\partial(\rho\boldsymbol{Y})}{\partial t} + \operatorname{div}(\rho\boldsymbol{Y} \otimes \boldsymbol{v}) &= \rho\mathfrak{f} \\
\frac{\partial(\rho Y)}{\partial t} + \operatorname{div}(\rho Y\boldsymbol{v}) &= \rho h
\end{aligned}$$

Mathematical entropy

$$\rho\widetilde{e_{\text{tot}}} \overset{\text{def}}{=} \rho e_{\text{tot}} + \frac{1}{2}\rho e_{\text{ref}} |\boldsymbol{Y}|^2$$

Mathematical entropy

$$\rho\widetilde{e_{\mathrm{tot}}} \overset{\mathsf{def}}{=} \rho e_{\mathrm{tot}} + \frac{1}{2}\rho e_{\mathrm{ref}}|\boldsymbol{Y}|^2$$

Additional balance law

$$\frac{\partial(\rho\widetilde{e_{\mathrm{tot}}})}{\partial t} + \mathrm{div}((\rho\widetilde{e_{\mathrm{tot}}}\boldsymbol{I} - \boldsymbol{T})\boldsymbol{v} + \boldsymbol{j}_e) = \rho\boldsymbol{f}\cdot\boldsymbol{v} + \rho e_{\mathrm{ref}}\boldsymbol{Y} : \mathfrak{f}$$

Mathematical entropy

$$\rho\widetilde{e_{\mathrm{tot}}} \overset{\mathsf{def}}{=} \rho e_{\mathrm{tot}} + \frac{1}{2}\rho e_{\mathrm{ref}}|\boldsymbol{Y}|^2$$

Additional balance law

$$\frac{\partial(\rho\widetilde{e_{\mathrm{tot}}})}{\partial t} + \mathrm{div}((\rho\widetilde{e_{\mathrm{tot}}}\boldsymbol{I} - \boldsymbol{T})\boldsymbol{v} + \boldsymbol{j}_e) = \rho\boldsymbol{f}\cdot\boldsymbol{v} + \rho e_{\mathrm{ref}}\,\boldsymbol{Y} : \mathfrak{f}$$

Strict convexity of $\rho\widetilde{e_{\mathrm{tot}}}$

$\rho\widetilde{e_{\mathrm{tot}}}$ is strictly convex with respect to $(\rho, \rho\eta, \rho\boldsymbol{j}_e, \rho\boldsymbol{v}, \rho\boldsymbol{F}, \rho\,\boldsymbol{Y}, \rho\,Y)$

$\Longleftrightarrow$ [Bouchut, 2004], [Wagner, 2009]

$\widetilde{e_{\mathrm{tot}}}$ is strictly convex with respect to $(\rho^{-1}, \eta, \boldsymbol{j}_e, \boldsymbol{v}, \boldsymbol{F}, \boldsymbol{Y}, Y)$

Mathematical entropy

$$\rho\widetilde{e_{\mathrm{tot}}} \overset{\text{def}}{=} \rho e_{\mathrm{tot}} + \frac{1}{2}\rho e_{\mathrm{ref}}|\boldsymbol{Y}|^2$$

Additional balance law

$$\frac{\partial(\rho\widetilde{e_{\mathrm{tot}}})}{\partial t} + \operatorname{div}((\rho\widetilde{e_{\mathrm{tot}}}\boldsymbol{I} - \boldsymbol{T})\boldsymbol{v} + \boldsymbol{j}_e) = \rho\boldsymbol{f}\cdot\boldsymbol{v} + \rho e_{\mathrm{ref}}\boldsymbol{Y} : \mathfrak{f}$$

Strict convexity of $\rho\widetilde{e_{\mathrm{tot}}}$

$\rho\widetilde{e_{\mathrm{tot}}}$ is strictly convex with respect to $(\rho, \rho\eta, \rho\boldsymbol{j}_e, \rho\boldsymbol{v}, \rho\boldsymbol{F}, \rho\boldsymbol{Y}, \rho Y)$

$\Longleftrightarrow$ [Bouchut, 2004], [Wagner, 2009]

$\widetilde{e_{\mathrm{tot}}}$ is strictly convex with respect to $(\rho^{-1}, \eta, \boldsymbol{j}_e, \boldsymbol{v}, \boldsymbol{F}, \boldsymbol{Y}, Y)$

$$\widetilde{e_{\mathrm{tot}}} = \frac{1}{2}|\boldsymbol{v}|^2 + e_{\mathrm{s}}\left(\rho, \eta - \frac{\alpha}{2}k_{\mathrm{B}}\log\left(\left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^2 Y^{\frac{1}{2}}\right)\right) + \frac{\tau_0}{2\kappa}|\boldsymbol{j}_e|^2 + \frac{\alpha}{2}K\operatorname{tr}(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^\top) + \frac{1}{2}e_{\mathrm{ref}}|\boldsymbol{Y}|^2$$

Strict convexity of $\widetilde{e_{\mathrm{tot}}}$ with respect to $(\rho^{-1}, \eta, \boldsymbol{j}_e, \boldsymbol{v}, \boldsymbol{F}, \boldsymbol{Y}, Y)$

$$\widetilde{e_{\mathrm{tot}}} = \frac{1}{2}|\boldsymbol{v}|^2 + e_{\mathrm{s}}\left(\rho, \eta - \frac{\alpha}{2} k_{\mathrm{B}} \log\left(\left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^2 Y^{\frac{1}{2}}\right)\right) + \frac{\tau_0}{2\kappa}|\boldsymbol{j}_e|^2 + \frac{\alpha}{2} K \operatorname{tr}(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^{\top}) + \frac{1}{2} e_{\mathrm{ref}}|\boldsymbol{Y}|^2$$

Strict convexity of $\widetilde{e_{\mathrm{tot}}}$ with respect to $(\rho^{-1}, \eta, \boldsymbol{j}_e, \boldsymbol{v}, \boldsymbol{F}, \boldsymbol{Y}, Y)$

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Strict convexity of $\widetilde{e_{\mathrm{tot}}}$ with respect to $(\rho^{-1}, \eta, \boldsymbol{j}_e, \boldsymbol{v}, \boldsymbol{F}, \boldsymbol{Y}, Y)$

$$\widetilde{e_{\mathrm{tot}}} = \frac{1}{2}|\boldsymbol{v}|^2 + e_{\mathrm{s}}\left(\rho, \eta - \frac{\alpha}{2} k_{\mathrm{B}} \log\left(\left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^2 Y^{\frac{1}{2}}\right)\right) + \frac{\tau_0}{2\kappa}|\boldsymbol{j}_e|^2 + \frac{\alpha}{2} K \operatorname{tr}(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^{\top}) + \frac{1}{2} e_{\mathrm{ref}} |\boldsymbol{Y}|^2$$

1. $\frac{\alpha}{2} K \operatorname{tr}(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^{\top})$ $\begin{cases} \text{convex with respect to } (\boldsymbol{F}, \boldsymbol{Y}) \text{ [Lieb, 1973]} \\ \text{strictly convex with respect to } \boldsymbol{F} \end{cases}$

Strict convexity of $\widetilde{e_{\mathrm{tot}}}$ with respect to $(\rho^{-1}, \eta, \boldsymbol{j}_e, \boldsymbol{v}, \boldsymbol{F}, \boldsymbol{Y}, Y)$

$$\widetilde{e_{\mathrm{tot}}} = \frac{1}{2}|\boldsymbol{v}|^2 + e_{\mathrm{s}}\left(\rho, \eta - \frac{\alpha}{2}k_{\mathrm{B}}\log\left(\left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^2 Y^{\frac{1}{2}}\right)\right) + \frac{\tau_0}{2\kappa}|\boldsymbol{j}_e|^2 + \frac{\alpha}{2}K\,\mathrm{tr}(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^\top) + \frac{1}{2}e_{\mathrm{ref}}|\boldsymbol{Y}|^2$$

1. $\frac{\alpha}{2}K\,\mathrm{tr}(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^\top)$ $\begin{cases}\text{convex with respect to } (\boldsymbol{F}, \boldsymbol{Y}) \text{ [Lieb, 1973]} \\ \text{strictly convex with respect to } \boldsymbol{F}\end{cases}$

Strict convexity of $\widetilde{e_{\text{tot}}}$ with respect to $(\rho^{-1}, \eta, \boldsymbol{j}_e, \boldsymbol{v}, \boldsymbol{F}, \boldsymbol{Y}, Y)$

$$\widetilde{e_{\text{tot}}} = \frac{1}{2}|\boldsymbol{v}|^2 + e_{\text{s}}\left(\rho, \eta - \frac{\alpha}{2} k_{\text{B}} \log\left(\left(\frac{\rho_{\text{R}}}{\rho}\right)^2 Y^{\frac{1}{2}}\right)\right) + \frac{\tau_0}{2\kappa}|\boldsymbol{j}_e|^2 + \frac{\alpha}{2} K \operatorname{tr}(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^{\top}) + \frac{1}{2} e_{\text{ref}} |\boldsymbol{Y}|^2$$

1. $\frac{\alpha}{2} K \operatorname{tr}(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^{\top})$ $\begin{cases} \text{convex with respect to } (\boldsymbol{F}, \boldsymbol{Y}) \text{ [Lieb, 1973]} \\ \text{strictly convex with respect to } \boldsymbol{F} \end{cases}$
2. $e_{\text{s}}\left(\rho, \eta - \frac{\alpha}{2} k_{\text{B}} \log\left(\left(\frac{\rho_{\text{R}}}{\rho}\right)^2 Y^{\frac{1}{2}}\right)\right)$ strictly convex with respect to (ρ^{-1}, η, Y)?

Strict convexity of $\widetilde{e_{\mathrm{tot}}}$ with respect to $(\rho^{-1}, \eta, \boldsymbol{j}_e, \boldsymbol{v}, \boldsymbol{F}, \boldsymbol{Y}, Y)$

$$\widetilde{e_{\mathrm{tot}}} = \frac{1}{2}|\boldsymbol{v}|^2 + e_{\mathrm{s}}\left(\rho, \eta - \frac{\alpha}{2} k_{\mathrm{B}} \log\left(\left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^2 Y^{\frac{1}{2}}\right)\right) + \frac{\tau_0}{2\kappa}|\boldsymbol{j}_e|^2 + \frac{\alpha}{2} K \operatorname{tr}(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^\top) + \frac{1}{2} e_{\mathrm{ref}} |\boldsymbol{Y}|^2$$

1. $\frac{\alpha}{2} K \operatorname{tr}(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^\top)$ $\begin{cases} \text{convex with respect to } (\boldsymbol{F}, \boldsymbol{Y}) \text{ [Lieb, 1973]} \\ \text{strictly convex with respect to } \boldsymbol{F} \end{cases}$
2. $e_{\mathrm{s}}\left(\rho, \eta - \frac{\alpha}{2} k_{\mathrm{B}} \log\left(\left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^2 Y^{\frac{1}{2}}\right)\right)$ strictly convex with respect to (ρ^{-1}, η, Y)?

   2.1 Polytropic gas equation of state

$$c_{\mathrm{V,s}}\theta_{\mathrm{ref}} \left(\frac{\rho}{\rho_{\mathrm{ref}}}\right)^{\gamma-1} \mathrm{e}^{\frac{\eta}{c_{\mathrm{V,s}}}} \left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^{-\frac{\alpha k_{\mathrm{B}}}{c_{\mathrm{V,s}}}} Y^{-\frac{\alpha k_{\mathrm{B}}}{4 c_{\mathrm{V,s}}}} \quad \sim \quad \frac{\mathrm{e}^{qy}}{x^p z^r}$$

Strict convexity of $\widetilde{e_{\mathrm{tot}}}$ with respect to $(\rho^{-1}, \eta, \boldsymbol{j}_e, \boldsymbol{v}, \boldsymbol{F}, \boldsymbol{Y}, Y)$

$$\widetilde{e_{\mathrm{tot}}} = \frac{1}{2}|\boldsymbol{v}|^2 + e_{\mathrm{s}}\left(\rho, \eta - \frac{\alpha}{2} k_{\mathrm{B}} \log\left(\left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^2 Y^{\frac{1}{2}}\right)\right) + \frac{\tau_0}{2\kappa}|\boldsymbol{j}_e|^2 + \frac{\alpha}{2} K \operatorname{tr}(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^{\top}) + \frac{1}{2} e_{\mathrm{ref}} |\boldsymbol{Y}|^2$$

1. $\frac{\alpha}{2} K \operatorname{tr}(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^{\top})$ $\begin{cases} \text{convex with respect to } (\boldsymbol{F}, \boldsymbol{Y}) \text{ [Lieb, 1973]} \\ \text{strictly convex with respect to } \boldsymbol{F} \end{cases}$
2. $e_{\mathrm{s}}\left(\rho, \eta - \frac{\alpha}{2} k_{\mathrm{B}} \log\left(\left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^2 Y^{\frac{1}{2}}\right)\right)$ strictly convex with respect to (ρ^{-1}, η, Y)?

   2.1 Polytropic gas equation of state

$$c_{\mathrm{V,s}}\theta_{\mathrm{ref}}\left(\frac{\rho}{\rho_{\mathrm{ref}}}\right)^{\gamma-1} \mathrm{e}^{\frac{\eta}{c_{\mathrm{V,s}}}} \left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^{-\frac{\alpha k_{\mathrm{B}}}{c_{\mathrm{V,s}}}} Y^{-\frac{\alpha k_{\mathrm{B}}}{4 c_{\mathrm{V,s}}}} \sim \frac{\mathrm{e}^{qy}}{x^p z^r}$$

   2.2 Noble–Abel stiffened-gas equation of state

$$c_{\mathrm{V,s}}\theta_{\mathrm{ref}}\left(\frac{\rho}{\rho_{\mathrm{ref}}}\frac{1}{1-b\rho}\right)^{\gamma-1} \mathrm{e}^{\frac{\eta}{c_{\mathrm{V,s}}}} \left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^{-\frac{\alpha k_{\mathrm{B}}}{c_{\mathrm{V,s}}}} Y^{-\frac{\alpha k_{\mathrm{B}}}{4 c_{\mathrm{V,s}}}} \sim \frac{\mathrm{e}^{qy}}{(x-b)^p x^r z^r}$$

Strict convexity of $\widetilde{e_{\mathrm{tot}}}$ with respect to $(\rho^{-1}, \eta, \boldsymbol{j}_e, \boldsymbol{v}, \boldsymbol{F}, \boldsymbol{Y}, Y)$

$$\widetilde{e_{\mathrm{tot}}} = \frac{1}{2}|\boldsymbol{v}|^2 + e_{\mathrm{s}}\left(\rho, \eta - \frac{\alpha}{2} k_{\mathrm{B}} \log\left(\left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^2 Y^{\frac{1}{2}}\right)\right) + \frac{\tau_0}{2\kappa}|\boldsymbol{j}_e|^2 + \frac{\alpha}{2} K \operatorname{tr}(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^\top) + \frac{1}{2} e_{\mathrm{ref}} |\boldsymbol{Y}|^2$$

1. $\frac{\alpha}{2} K \operatorname{tr}(\boldsymbol{F}\boldsymbol{Y}^{-\frac{1}{2}}\boldsymbol{F}^\top)$ $\begin{cases} \text{convex with respect to } (\boldsymbol{F}, \boldsymbol{Y}) \text{ [Lieb, 1973]} \\ \text{strictly convex with respect to } \boldsymbol{F} \end{cases}$
2. $e_{\mathrm{s}}\left(\rho, \eta - \frac{\alpha}{2} k_{\mathrm{B}} \log\left(\left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^2 Y^{\frac{1}{2}}\right)\right)$ strictly convex with respect to (ρ^{-1}, η, Y)?

2.1 Polytropic gas equation of state

$$c_{\mathrm{V,s}}\theta_{\mathrm{ref}}\left(\frac{\rho}{\rho_{\mathrm{ref}}}\right)^{\gamma-1} \mathrm{e}^{\frac{\eta}{c_{\mathrm{V,s}}}} \left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^{-\frac{\alpha k_{\mathrm{B}}}{c_{\mathrm{V,s}}}} Y^{-\frac{\alpha k_{\mathrm{B}}}{4 c_{\mathrm{V,s}}}} \sim \frac{\mathrm{e}^{qy}}{x^p z^r}$$

2.2 Noble–Abel stiffened-gas equation of state

$$c_{\mathrm{V,s}}\theta_{\mathrm{ref}}\left(\frac{\rho}{\rho_{\mathrm{ref}}}\frac{1}{1-b\rho}\right)^{\gamma-1} \mathrm{e}^{\frac{\eta}{c_{\mathrm{V,s}}}} \left(\frac{\rho_{\mathrm{R}}}{\rho}\right)^{-\frac{\alpha k_{\mathrm{B}}}{c_{\mathrm{V,s}}}} Y^{-\frac{\alpha k_{\mathrm{B}}}{4 c_{\mathrm{V,s}}}} \sim \frac{\mathrm{e}^{qy}}{(x-b)^p x^r z^r}$$

Local well-posedness of the model

- Governing equations for $(\rho, \rho\eta, \rho\boldsymbol{j}_e, \rho\boldsymbol{v}, \rho\boldsymbol{F}, \rho\boldsymbol{Y}, \rho Y)$:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\boldsymbol{v}) &= 0 \\
\frac{\partial(\rho\eta)}{\partial t} + \operatorname{div}\left(\rho\eta\boldsymbol{v} + \frac{\boldsymbol{j}_e}{\theta}\right) &= \xi \\
\frac{\partial(\rho\boldsymbol{j}_e)}{\partial t} + \operatorname{div}\left(\rho\boldsymbol{j}_e \otimes \boldsymbol{v} + \frac{\kappa}{\tau_0}(\log\theta)\boldsymbol{I}\right) &= -\frac{\boldsymbol{j}_e}{\tau_0\theta} \\
\frac{\partial(\rho\boldsymbol{v})}{\partial t} + \operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v} - \boldsymbol{T}) &= \rho\boldsymbol{f} \\
\frac{\partial(\rho\boldsymbol{F})}{\partial t} + \operatorname{div}\left(\rho\left(\boldsymbol{F}\otimes\boldsymbol{v} - \boldsymbol{v}\otimes\boldsymbol{F}^{\top}\right)\right) &= \boldsymbol{0} \\
\frac{\partial(\rho\boldsymbol{Y})}{\partial t} + \operatorname{div}(\rho\boldsymbol{Y}\otimes\boldsymbol{v}) &= \rho\mathfrak{f} \\
\frac{\partial(\rho Y)}{\partial t} + \operatorname{div}(\rho Y\boldsymbol{v}) &= \rho h
\end{aligned}
$$

- Additional balance law for **strictly convex** $\rho\widetilde{e_{\text{tot}}}$

$$
\frac{\partial(\rho\widetilde{e_{\text{tot}}})}{\partial t} + \operatorname{div}((\rho\widetilde{e_{\text{tot}}}\boldsymbol{I} - \boldsymbol{T})\boldsymbol{v} + \boldsymbol{j}_e) = \rho\boldsymbol{f}\cdot\boldsymbol{v} + \rho e_{\text{ref}}\boldsymbol{Y} : \mathfrak{f}
$$

Local well-posedness of the model

Theorem (Polytropic gas equation of state)

Let $\mathcal{U}$ be an open subset of the convex set

$$\mathcal{V} \overset{\text{def}}{=} (0, +\infty) \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^9 \times \mathbb{R}^6_{\text{sym},>} \times (0, +\infty).$$

Consider the system of balance laws governing the motion of compressible heat-conducting Maxwell viscoelastic fluid with the specific internal energy of the solvent contribution given by the polytropic gas equation of state. Assume that $\boldsymbol{u}_0 \in \mathcal{U}$ and $\widetilde{\boldsymbol{u}_0} \in H^s(\mathbb{R}^3; \mathbb{R}^{24})$ with $s > \frac{5}{2}$ such that $\boldsymbol{u}_0 + \widetilde{\boldsymbol{u}_0}$ is compactly supported in $\mathcal{U}$.

Then, there exists $T > 0$ and a unique classical solution $\boldsymbol{u} \in C^1(\mathbb{R}^3 \times [0, T]; \mathcal{U})$ of the Cauchy problem associated with our system of balance laws and initial data $\boldsymbol{u}(0) = \boldsymbol{u}_0 + \widetilde{\boldsymbol{u}_0}$. Furthermore, $\boldsymbol{u} - \boldsymbol{u}_0 \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$.

Local well-posedness of the model

Theorem (Noble–Abel stiffened-gas equation of state)

Let $\mathcal{U}$ be an open subset of the convex set

$$\mathcal{V} \overset{\text{def}}{=} (0, b) \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^9 \times \mathbb{R}^6_{\text{sym},>} \times (0, +\infty).$$

Consider the system of balance laws governing the motion of compressible heat-conducting Maxwell viscoelastic fluid with the specific internal energy of the solvent contribution given by the Noble–Abel stiffened-gas equation of state. Assume that $\boldsymbol{u}_0 \in \mathcal{U}$ and $\widetilde{\boldsymbol{u}_0} \in H^s(\mathbb{R}^3; \mathbb{R}^{24})$ with $s > \frac{5}{2}$ such that $\boldsymbol{u}_0 + \widetilde{\boldsymbol{u}_0}$ is compactly supported in $\mathcal{U}$.

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Consistency with the physical interpretation

Initial conditions

$$(\rho_0, \rho_0\eta_0, \rho_0\boldsymbol{j}_{e0}, \rho_0\boldsymbol{v}_0, \rho_0\boldsymbol{F}_0, \rho_0\,\boldsymbol{Y}_0, \rho_0\,Y_0)$$

need to satisfy for $\boldsymbol{x} \in \mathbb{R}^3$

- $Y_0(\boldsymbol{x}) = \det(\boldsymbol{Y}_0^{-1}(\boldsymbol{x}))$,
- $\rho_\mathrm{R} = \rho_0(\boldsymbol{x}) \det \boldsymbol{F}_0(\boldsymbol{x})$,
- $\boldsymbol{F}_0(\boldsymbol{x}) = \frac{\partial \boldsymbol{\chi}_0}{\partial \boldsymbol{X}}(\boldsymbol{\chi}_0^{-1}(\boldsymbol{x}))$ for some invertible $\boldsymbol{\chi}_0 : \mathbb{R}^3 \to \mathbb{R}^3$.

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We need to show that the solution

$$(\rho, \rho\eta, \rho\boldsymbol{j}_e, \rho\boldsymbol{v}, \rho\boldsymbol{F}, \rho\,\boldsymbol{Y}, \rho\,Y)$$

satisfies for $t \in (0, T]$ and $\boldsymbol{x} \in \mathbb{R}^3$

- $Y(\boldsymbol{x}, t) = \det(\boldsymbol{Y}^{-1}(\boldsymbol{x}, t))$,
- $\rho_{\mathrm{R}} = \rho(\boldsymbol{x}, t) \det \boldsymbol{F}(\boldsymbol{x}, t)$,
- $\boldsymbol{F}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{\chi}_t}{\partial \boldsymbol{X}}(\boldsymbol{\chi}_t^{-1}(\boldsymbol{x}))$ for some invertible $\boldsymbol{\chi}_t : \mathbb{R}^3 \to \mathbb{R}^3$,
- $\boldsymbol{v}(x, t) = \frac{\partial \boldsymbol{\chi}_t}{\partial t}(\boldsymbol{\chi}_t^{-1}(\boldsymbol{x}))$.

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- $Y(\boldsymbol{x}, t) = \det(\boldsymbol{Y}^{-1}(\boldsymbol{x}, t)) \Longleftarrow$ uniqueness of the solution,
- $\rho_{\mathrm{R}} = \rho(\boldsymbol{x}, t) \det \boldsymbol{F}(\boldsymbol{x}, t)$,
- $\boldsymbol{F}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{\chi}_t}{\partial \boldsymbol{X}}(\boldsymbol{\chi}_t^{-1}(\boldsymbol{x}))$ for some invertible $\boldsymbol{\chi}_t : \mathbb{R}^3 \to \mathbb{R}^3$,
- $\boldsymbol{v}(x, t) = \frac{\partial \boldsymbol{\chi}_t}{\partial t}(\boldsymbol{\chi}_t^{-1}(\boldsymbol{x}))$.

Alternative model formulation (motivated by [Wagner, 2009])

Helmholtz free energy:

$$\psi(\rho, \theta, \boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{j}_e) = \psi_{\mathrm{s}}(\rho, \theta) + \frac{\tau_0}{2\kappa}|\boldsymbol{j}_e|^2 + \frac{\alpha}{2}(K_1 \operatorname{tr} \boldsymbol{C}_1 - k_{\mathrm{B}}\theta \log\det \boldsymbol{C}_1) + \frac{\alpha}{2}(K_2 \operatorname{tr} \boldsymbol{C}_2 - k_{\mathrm{B}}\theta \log\det \boldsymbol{C}_2)$$

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Objective time rates for the conformation tensors:

$$\overset{\nabla}{\boldsymbol{C}} \overset{\text{def}}{=} \frac{\mathrm{d}\boldsymbol{C}_1}{\mathrm{d}t} - (\nabla \boldsymbol{v})\boldsymbol{C}_1 - \boldsymbol{C}_1(\nabla \boldsymbol{v})^\top$$

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- short-time ∃! of the classical solution of the associated Cauchy problem
- consistency with the physical interpretation [Wagner, 1994], [Wagner, 2009]

Conclusions

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- To ensure consistency with the physical interpretation we have proposed an alternative model formulation of the standard Maxwell model.

Benzoni-Gavage, S. and Serre, D. (2006).
Multi-dimensional hyperbolic partial differential equations: First-order systems and applications.
Oxford Mathematical Monographs. OUP Oxford.

Bernstein, B., Kearsley, E., and Zapas, L. (1964).
Thermodynamics of perfect elastic fluids.
Journal of Research of the National Bureau of Standards–B. Mathematics and Mathematical Physics, 68B, pages 103–113.

Bernstein, B., Kearsley, E. A., and Zapas, L. J. (1963).
A study of stress relaxation with finite strain.
Trans. Soc. Rheol., 7(1):391–410.

Bouchut, F. (2004).
Nonlinear Stability of Finite Volume Methods for Hyperbolic Conservation Laws and Well-Balanced Schemes for Sources, volume 2/2004.
Birkhäuser.

Boyaval, S. (2020).
Viscoelastic flows of Maxwell fluids with conservation laws.
working paper or preprint.

Cattaneo, C. (2011).
Sulla Conduzione Del Calore, pages 485–485.
Springer Berlin Heidelberg, Berlin, Heidelberg.

References ii

Chandrasekharaiah, D. S. (1986).
Thermoelasticity with Second Sound: A Review.
Applied Mechanics Reviews, 39(3):355–376.

Dressler, M., Edwards, B. J., and Öttinger, H. C. (1999).
Macroscopic thermodynamics of flowing polymeric liquids.
Rheol. Acta, 38(2):117–136.

Jou, D., Casas-Vázquez, J., and Lebon, G. (1999).
Extended irreversible thermodynamics revisited (1988-98).
Reports on Progress in Physics, 62(7):1035–1142.

Kaye, A. (1962).
Non-Newtonian flow in incompressible fluids.
Technical report, College of Aeronautics Cranfield.

Le Métayer, O. and Saurel, R. (2016).
The Noble–Abel stiffened-gas equation of state.
Phys. Fluids, 28(4):046102.

Lieb, E. H. (1973).
Convex trace functions and the wigner-yanase-dyson conjecture.
Advances in Mathematics, 11(3):267 – 288.

References iii

Wagner, D. (1994).
Conservation laws, coordinate transformations, and differential forms.
In Glimm, J., M.J., G., J.W., G., and B.J., P., editors, *Hyperbolic Problems: Theory, Numerics, Applications*, pages 471–477. World Scientific.

Wagner, D. H. (2009).
Symmetric-hyperbolic equations of motion for a hyperelastic material.
Journal of Hyperbolic Differential Equations, 06(03):615–630.