

Recent studies on the image domain of starlike functions

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Background

Let \mathbb{C} be the complex plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in \mathbb{C} . A function f is analytic at a point $z_0 \in D$ if it is differentiable in some neighbourhood of z_0 and it is analytic in a domain D if it is analytic at all points in domain D .

Definition

An analytic function f is called univalent in a domain D if it does not take the same value twice, so that for $z_1, z_2 \in D$,

$$f(z_1) \neq f(z_2) \text{ for } z_1 \neq z_2.$$

Geometrically, this means that different points in the domain will be mapped into different points on the image domain.

One of the most basic results in the theory of univalent functions in one variable is the Riemann mapping theorem. Its failure in several variables is one of the key differences between complex analysis in one variable and higher dimensions.

Riemann Mapping Theorem

Every simply connected domain D which is a proper subset of \mathbb{C} can be mapped conformally onto the unit disc. Moreover, if $z_0 \in D$, there is a unique conformal map of D onto \mathbb{D} such that $f(z_0) = 0$ and $f'(z_0) > 0$.

The theory of univalent functions is so vast and complicated that certain simplifying assumptions are necessary. The most obvious one is to replace the arbitrary domain D by one that is convenient and the most attractive selection is the open unit disc \mathbb{D} .

For example, consider the function $f(z) = (1 + z)^2$ in the open unit disc \mathbb{D} . The univalence of this function in \mathbb{D} is easy to see on geometric grounds. Indeed $1 + z$ shifts the open unit disc to the right and the effect of squaring $1 + z$ is easy to visualize. The same type of argument shows that $w = (1 + z)^3$ is not univalent in \mathbb{D} .

Let \mathcal{A} represent the class of functions f of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots = z + \sum_{n=2}^{\infty} a_nz^n, \quad (1)$$

which are analytic in the open unit disc \mathbb{D} and normalized under the conditions

$$f(0) = f'(0) - 1 = 0.$$

Moreover, by \mathcal{S} we shall represent the class of all functions in \mathcal{A} which are univalent in \mathbb{D} .

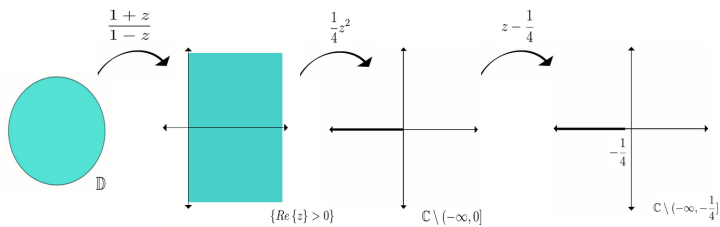
The function

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n \quad (z \in \mathbb{D})$$

is an element of \mathcal{S} and named *Koebe function*. It maps \mathbb{D} onto the complex plane except for a slit along the half-line $(-\infty, -\frac{1}{4}]$ and is univalent. This is the best seen by writing

$$k(z) = \frac{1}{4} \left(\frac{1+z}{1-z} \right)^2 - \frac{1}{4}$$

and observe that the function $(1+z)/(1-z)$ maps \mathbb{D} conformally onto the right half-plane $\Re(w) > 0$.



Koebe initiated in 1907 [15] the study about univalent functions, while Bieberbach presented in 1916 would soon become a famous conjecture. In 1916, Bieberbach [3] conjectured that for $f \in \mathcal{S}$,

$$|a_n| \leq n \quad (n \geq 2).$$

He proved only for the case when $n = 2$. For many years this conjecture has remained as a open problem for the mathematicians and has inspired the development of several remarkable techniques in the field. In 1985, Louis de Branges [5] proved the Bieberbach's conjecture for all the coefficients n . Although almost 70 years had passed until the Bieberbach conjecture was finally proved in this paper, bounds for the Taylor coefficients were obtained in the meantime for some subclasses of univalent functions. After the proof of the Bieberbach conjecture, the study of different subclasses of analytic and univalent functions have began to take shape, still remaining an interesting subject.

Next, the well known class of functions with positive real part, consisting of all functions p analytic in \mathbb{D} satisfying $p(0) = 1$ and $\Re p(z) > 0$, is usually denoted by \mathcal{P} and called the Carathéodory class. Each $p \in \mathcal{P}$ has a Taylor series expansion

$$p(z) = 1 + x_1z + x_2z^2 + x_3z^3 + \cdots \quad (x_1 > 0)$$

with coefficients satisfying $|x_n| \leq 2$ for $n \in \mathbb{N}$ (see [20]). More refinement coefficients bounds in the Carathéodory class was obtained by Grenander and Szegő [10].

For example, the function

$$p(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{D})$$

belongs to \mathcal{P} . This function presents a conformal map of \mathbb{D} onto the right-half plane and consequently it plays a fundamental role in \mathcal{P} , similar to the Koebe function in \mathcal{S} . Indeed, p is not required to be univalent. Thus $p(z) = 1 + z^n$ is in \mathcal{P} for any integer $n \geq 0$, but if $n \geq 2$, this function is not univalent.

An analytic function f_1 is subordinate to an analytic function f_2 , written

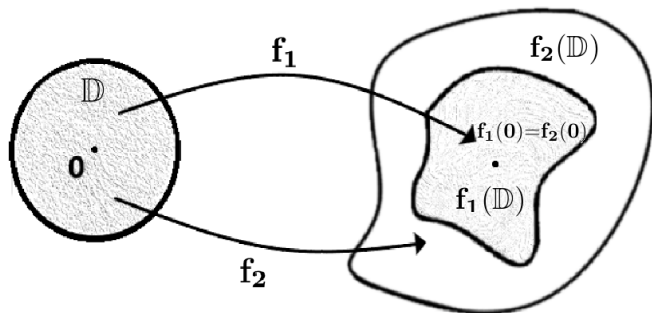
$$f_1(z) \prec f_2(z), \quad (z \in \mathbb{D})$$

provided there is an analytic function w , defined on \mathbb{D} with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \quad (z \in \mathbb{D})$$

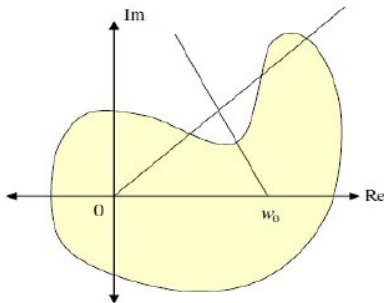
such that

$$f_1(z) = f_2(w(z)) \quad (z \in \mathbb{D}).$$



Definition

A set D in the plane is said to be starlike respect to w_0 an interior point of D if each ray with initial point w_0 intersects the interior of D in a set that is either a line segment or a ray. If a function $f(z)$ maps \mathbb{D} onto a domain that is starlike with respect to w_0 , then we say that $f(z)$ is starlike with respect to w_0 . In the special case $w_0 = 0$, we say that $f(z)$ is starlike functions.



We indicate this set of functions by \mathcal{S}^* . The class \mathcal{S}^* of starlike functions is a collection of functions $f \in \mathcal{S}$ for which

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{D}).$$

Without doubt, since the Koebe function is an element of the class \mathcal{S}^* , it is the most studied subclass of univalent functions. The image domain of starlike functions has followed in a rich set of properties for functions in \mathcal{S}^* , some of which are true for the wider class \mathcal{S} , while other are open problems.

Now it is a simple matter to conclude that there is a relation between the classes \mathcal{S}^* and \mathcal{P} .

Theorem

$f \in \mathcal{S}^*$ if and only if

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}.$$

This implication reveals that information about the class of \mathcal{S}^* can be drawn whenever the properties of functions in the class \mathcal{P} are known. For example, assume that the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

fulfills $\Re(f'(z)) > 0$. Then $f'(z) = p(z) \in \mathcal{P}$ and so $a_n = \frac{x_n}{n}$.

Therefore, if the bound on the coefficients of Carathéodory functions are known, then estimate on $|a_n|$ can be found easily.

Recent studies

The relation of subordination is used to establish many classes of functions encountered in the Theory of Geometric Functions and to investigate the properties of these classes. Let us recall

$$S^*[\psi] := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \psi(z), z \in \mathbb{D} \right\}, \quad (2)$$

where ψ is an analytic function in \mathbb{D} with $\psi(0) = 1$. For $\psi(z) = \frac{1+z}{1-z}$, one can obtain the well-known class S^* of starlike.

Many researchers have defined different classes of functions by using other functions instead of the ψ function in (2). When the figure of the unit circle under these functions is calculated, very interesting results are obtained. In [21], Robertson showed that the figure of the unit circle is $\Re(w) > \gamma$ using $\psi(z) = \frac{1+(1-2\gamma)z}{1-z}$. In this case, the set (2) becomes the class $S^*(\gamma)$ of starlike functions of order γ . Janowski obtained that $\psi(\mathbb{D})$ is a disc in [11]. By taking $\psi(z) = \left(\frac{1+z}{1-z}\right)^\beta$ ($0 < \beta \leq 1$), the class of strongly starlike functions of order β is defined by Stankiewicz in [27] and demonstrated that the figure of unit circle under this class is an angle.

With different choices of ψ , it is obtained that $\psi(\mathbb{D})$ is parabola in [18], is ellipse and hyperbola in [12, 13].

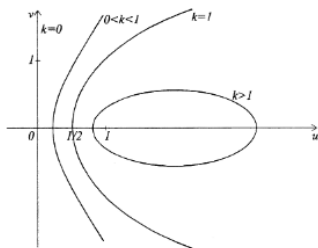


Fig. 1.

Note that, by (2.3), Ω_k is a domain such that, $1 \in \Omega_k$ and $\partial\Omega_k$ is a curve defined by the equality

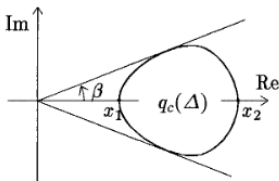
$$\partial\Omega_k = \{w = u + iv: u^2 = k^2(u-1)^2 + k^2v^2\}, \quad (0 \leq k < \infty). \quad (2.2)$$

If $\psi(z) = \sqrt{1+z}$ is chosen in order that $\sqrt{1} = 1$, then the figure of $\psi(\mathbb{D})$ is interior of the right side of the Lemniscate. Obtained class is considered in [22], [23] by Sokol et al.

Let us introduce the class

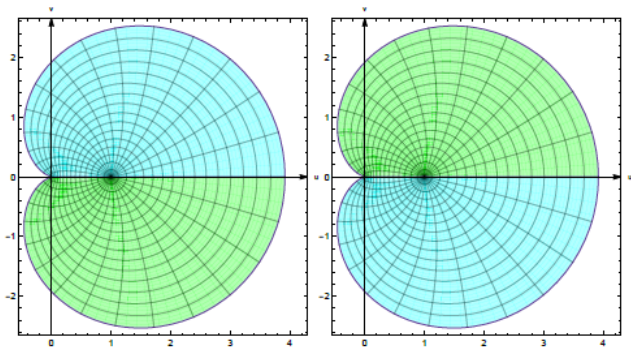
$$\mathcal{S}^*(q_c) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec q_c(z), \right. \\ \left. q_c(z) = \sqrt{cz+1}, q_c(0) = 1, z \in \Delta \right\}, \quad c \in (0, 1].$$

Notice that for $c \in (0, 1)$ the set $q_c(\Delta) = \{w \in \mathbb{C} : \operatorname{Re} w > 0 \wedge |w^2 - 1| < c\}$ is the interior of the right half of the Cassini's curve $|w^2 - 1| = c$ and in the special case $c = 1$ this curve is the Bernoulli's lemniscate; see [5], pp. 231–235.



$$x_1 = \sqrt{1-c}, \\ x_2 = \sqrt{1+c}, \\ \beta = \frac{1}{2} \arcsin c.$$

Further, Masih and Kanas [17] introduced and studied the classes $S_L[\psi]$ and $K_L[\psi]$ ($\psi(z) = (1 + \chi z)^2$, $0 < \chi \leq 1/\sqrt{2}$) associated with the limaçon.



The image of D under $L_s(z)$

The classes $S_N[\psi]$ and $K_N[\psi]$ ($\psi(z) = 1 + z - z^3/3$) endowed with a nephroid domain were presented by Wani and Swaminathan [29].

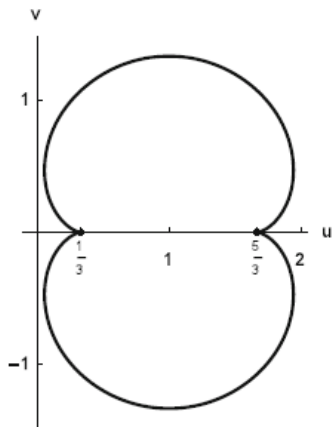
Remark 1

The function $\varphi_{Ne}(z) = 1 + z - z^3/3$ maps \mathbb{D} onto the region bounded by the nephroid

$$\left((u-1)^2 + v^2 - \frac{4}{9} \right)^3 - \frac{4v^2}{3} = 0,$$

which is symmetric about the real axis and lies completely inside the right-half plane $u > 0$.

3.1.5 Nephroid: the boundary
curve of $\varphi_{Nc}(\mathbb{D})$, where
 $\varphi_{Nc}(z) = 1 + z - z^3/3$



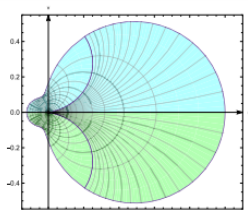
In a recent paper, Kanas and Masih [14] introduced the Pascal snail:

For $-1 \leq \alpha \leq 1$, $-1 \leq \beta \leq 1$, $\alpha\beta \neq \pm 1$ and $0 \leq \gamma < 1$ let $\mathcal{L}_{\alpha,\beta,\gamma}$ denote the complex valued mapping

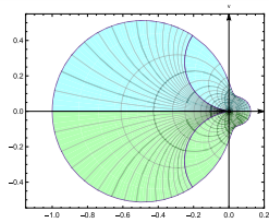
$$\begin{aligned} \mathcal{L}_{\alpha,\beta,\gamma} f(z) &= \frac{(2-2\gamma)z}{(1-\alpha z)(1-\beta z)} \\ &= \sum_{n=1}^{\infty} B_n z^n = \begin{cases} (2-2\gamma) \sum_{n=1}^{\infty} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) z^n & \alpha \neq \beta \\ (2-2\gamma) \sum_{n=1}^{\infty} n \alpha^{n-1} z^n & \alpha = \beta \end{cases}, \end{aligned}$$

where $z \in \mathbb{D}$. We note that $\mathcal{L}_{\alpha,\beta,\gamma}$ maps \mathbb{D} onto a domain $D(\alpha, \beta, \gamma)$ whose boundary is given by

$$\partial D(\alpha, \beta, \gamma) = \left\{ w = u + iv : \frac{[2(1-\gamma)u + (\alpha+\beta)(u^2+v^2)]^2}{(1+\alpha\beta)^2} + \frac{4(1-\gamma)^2 v^2}{(1-\alpha\beta)^2} - (u^2 + v^2)^2 = 0 \right\}.$$

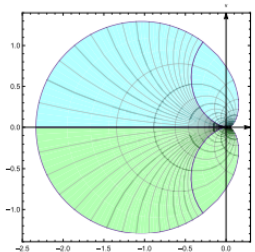


(A) $\alpha = -0.4, \beta = 0.9, \gamma = 0.93$

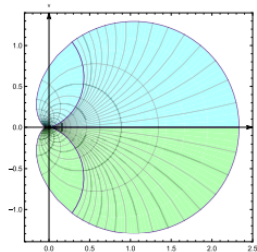


(B) $\alpha = -0.9, \beta = 0.4, \gamma = 0.93$

The image of \mathbb{D} under $\Sigma_{\alpha, \beta, \gamma}(z)$



(A) $\alpha = -0.4, \beta = -0.9, \gamma = 0.93$



(B) $\alpha = 0.9, \beta = 0.4, \gamma = 0.93$

- $\mathcal{S}_L^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, z \in \mathbb{D} \right\},$
- $\mathcal{S}_p^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, z \in \mathbb{D} \right\},$
- $\mathcal{S}_e^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^z, z \in \mathbb{D} \right\},$
- $\mathcal{S}_c^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, z \in \mathbb{D} \right\},$

- $\mathcal{S}_{\sin}^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \sin z, z \in \mathbb{D} \right\},$
- $\mathcal{S}_{\mathbb{Q}}^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2}, z \in \mathbb{D} \right\},$
- $\mathcal{S}_R^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{z}{k} \left(\frac{k+z}{k-z} \right), k = \sqrt{2} + 1, z \in \mathbb{D} \right\},$
- $\mathcal{S}_{RL}^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{2} - (\sqrt{2} - 1) \sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}, z \in \mathbb{D} \right\}.$

On the other hand, Yılmaz Özgür and Sokół [28] studied this topic from a different viewpoint.

Definition

Let k be any positive real number. The function $f \in \mathcal{S}$ belongs to the class \mathcal{SL}^k if satisfies the condition that

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}_k(z),$$

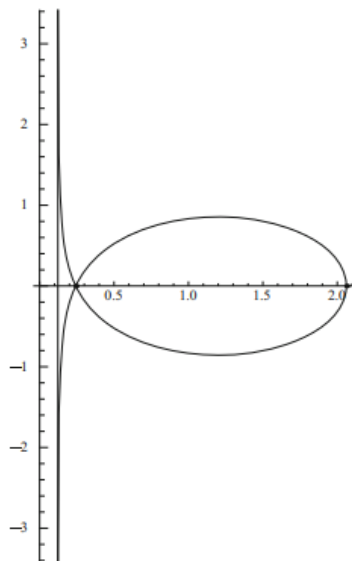
where

$$\tilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - (\tau_k^2 - 1)z - \tau_k^2 z^2}, \quad \tau_k = \frac{k - \sqrt{4 + k^2}}{2}, \quad z \in \mathbb{D}.$$

The image of unit circle of $\tilde{p}_k(z)$ is the curve \mathcal{C}_k with equation

$$x = \frac{k\sqrt{k^2 + 4}}{2[k^2 + 2 - 2\cos\theta]}, \quad y = \frac{(4\cos\theta - k^2)\sin\theta}{2[k^2 + 2 - 2\cos\theta](1 + \cos\theta)}, \quad \theta \in [0, 2\pi) \setminus \{\pi\}.$$

The curve C_k for $k = \frac{1}{2}$







Motivated by the work of Yılmaz Özgür and Sokół, we study the function





$$\tilde{j}_k(z) = \frac{1 + \delta_k^2 z^2}{1 - k\delta_k z - 2\delta_k^2 z^2}, \quad \delta_k = \frac{k - \sqrt{8 + k^2}}{2}, \quad z \in \mathbb{D},$$

where k is any positive real number [1].





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