Rigid Manifolds via Product Quotients

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Definition

Let X be a compact complex manifold, then

1) X is said to be (locally) rigid, if for each deformation of X,

$$f\colon (\mathfrak{X}, X) \to (\mathcal{B}, b_0)$$

   there is an open neighbourhood $U \subset \mathcal{B}$ of b_0 such that $X_t \simeq X$ for all $t \in U$.

2) X is said to be infinitesimally rigid if $H^1(X, \Theta_X) = 0$, where Θ_X is the sheaf of holomorphic vector fields on X.

Remark

- 2) $\Rightarrow$ 1) *(Kodaira-Spencer-Kuranishi)*
- 1) $\not\Rightarrow$ 2) *example by Bauer Pignatelli 2019, further examples by Böhning von Bothmer and Pignatelli 2020*

- The only rigid curve is $\mathbb{P}^1$
- Bauer Catanese 2018:
  - If X is a rigid surface, then $\kappa(X) = 2$ or $\kappa(X) = -\infty$.
  - For all $n \geq 3$ and for all $2 \leq \kappa \leq n$ there exists a rigid n-fold $X_{n,\kappa}$ with $\kappa(X_{n,\kappa}) = \kappa$.
  - For all $n \geq 4$ there exists a rigid n-fold X_n with $\kappa(X_n) = 0$.
  - Observation: All known examples of rigid manifolds of general type are $K(\pi, 1)$ i.e. they have contractible universal cover.

Motivation

The questions:

A) Do rigid manifolds X_n of dimension $n \geq 3$ with $\kappa(X_n) = 1$ exist?

B) Is the universal cover of a rigid manifold of general type always contractible?

Our results:

Theorem A) (2019 Bauer -)

For all $n \geq 3$ there exists a rigid n-fold X_n with $\kappa(X_n) = 1$.

Theorem B) (2021 Frapporti -)

For all $n \geq 3$ there exists a rigid n-fold X_n of general type with a non-contractible universal cover. (Surface case is still open.)

A) Rigid manifolds with Kodaira dimension one

We consider

- $C := \{x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0 = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$ Klein quartic
- $E = \mathbb{C}/\mathbb{Z}[\zeta_3]$ Fermat elliptic curve
- $G := \langle s, t \mid s^3 = 1,\ t^7 = 1,\ sts^{-1} = t^4 \rangle$

- There is a G-action ϕ on E defined by

$$s(z) = \zeta_3 z \quad \text{and} \quad t(z) = z + \frac{1 + 3\zeta_3}{7}$$

  on E and
- an action ψ on C by

$$s = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad t = \begin{pmatrix} \zeta_7^4 & 0 & 0 \\ 0 & \zeta_7^2 & 0 \\ 0 & 0 & \zeta_7 \end{pmatrix}$$

  on C.
- We define $Y_n := (C \times E^{n-1})/G$ with respect to the product action $\psi \times \phi^{n-1}$.

A) Rigid manifolds with Kodaira dimension one

Proposition

The quotient $Y_n := (C \times E^{n-1})/G$ is singular with singularities

$$3^{n-1} \times \frac{1}{3}(1, \dots, 1), \qquad 3^{n-1} \times \frac{1}{3}(1, \dots, 1, 2) \qquad (*)$$

These singularities are isolated and canonical $\Longrightarrow$ $\kappa(X_n) = 1$ for any resolution $\rho \colon X_n \to Y_n$.

Proposition

I) It holds $H^1(\Theta_{C \times E^{n-1}})^G = 0 \quad \Longrightarrow \quad H^1(\Theta_{Y_n}) \simeq H^1(\Theta_{C \times E^{n-1}})^G = 0$.

II) Let U be any of $(*)$ then there exists a resolution $\rho \colon \widehat{U} \to U$ with

$$\rho_* \Theta_{\widehat{U}} = \Theta_U \qquad \text{and} \qquad R^1 \rho_* \Theta_{\widehat{U}} = 0.$$

A) Rigid manifolds with Kodaira dimension one

Assuming Proposition I) and II) we get

Conclusion

- Singularities are isolated $\implies$ resolutions glue $\rho\colon X_n \to Y_n$ with same properties.
- The low terms of Leray's spectral sequence give

$$H^1(\Theta_{X_n}) \simeq H^1(\Theta_{Y_n}) \simeq H^1(\Theta_{C\times E^{n-1}})^G = 0$$

- We conclude that X_n is an infinitesimally rigid manifold of Kodaira dimension 1.

We need to prove the Proposition.

I) Sketch of proof of $H^1(\Theta_{C\times E^{n-1}})^G = 0$

- We write $T := E_1 \times \ldots \times E_{n-1} \times C$, where $E_i = E$.
- $\Theta_T := p_1^*\Theta_{E_1} \oplus \ldots \oplus p_{n-1}^*\Theta_{E_{n-1}} \oplus p_n^*\Theta_C$,

$$\implies H^1(\Theta_T) = H^1(p_1^*\Theta_{E_1}) \oplus \ldots \oplus H^1(p_{n-1}^*\Theta_{E_{n-1}}) \oplus H^1(p_n^*\Theta_C),$$

A) Rigid manifolds with Kodaira dimension one

- Künneth formula:
  - $H^1(p_i^*\Theta_{E_i}) = H^1(\Theta_{E_i}) \oplus H^0(\Theta_{E_i}) \otimes \left[\bigoplus_{i \neq j} H^1(\mathcal{O}_{E_j}) \oplus H^1(\mathcal{O}_C)\right]$
  - $H^1(p_n^*\Theta_C) = H^1(\Theta_C)$, because $H^0(\Theta_C) = 0$.

Rigidity conditions:

- $H^1(\Theta_C)^G = H^1(\Theta_{E_i})^G = 0$ and
- $\left[H^0(\Theta_{E_i}) \otimes H^1(\mathcal{O}_{E_j})\right]^G = \left[H^0(\Theta_{E_i}) \otimes H^1(\mathcal{O}_C)\right]^G = 0.$

- $H^1(\Theta_C)^G = H^1(\Theta_{E_i})^G = 0$, holds because

$$C \to C/G \simeq \mathbb{P}^1 \quad \text{and} \quad E_i \to E_i/G \simeq \mathbb{P}^1$$

are branched in three points.

Rigidity conditions

- We verify $\left[H^0(\Theta_{E_i}) \otimes H^1(\mathcal{O}_{E_j})\right]^G = \left[H^0(\Theta_{E_i}) \otimes H^1(\mathcal{O}_C)\right]^G = 0$, using the representations

$$\phi_C \colon G \to GL(H^0(K_C)) \quad \text{and} \quad \phi_{E_i} \colon G \to GL(H^0(K_{E_i}))$$

and Serre duality.

II) Resolutions

For each of the singularities

$$\frac{1}{3}(1, \ldots, 1) \qquad \text{and} \qquad \frac{1}{3}(1, \ldots, 1, 2)$$

we need to construct a resolution $\rho \colon \widehat{U} \to U$ with

$$\rho_* \Theta_{\widehat{U}} = \Theta_U \qquad \text{and} \qquad R^1 \rho_* \Theta_{\widehat{U}} = 0.$$

A) Rigid manifolds with Kodaira dimension one

Remark

A cyclic quotient singularity $\frac{1}{m}(a_1, \ldots, a_n)$ is an affine toric variety, defined by

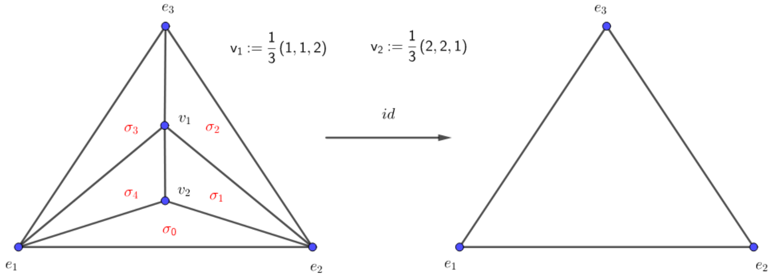
$$N := \mathbb{Z}^n + \frac{\mathbb{Z}}{m}(a_1, \ldots, a_n) \qquad \text{and} \qquad \sigma := cone(e_1, \ldots, e_n).$$

- We use the powerful tools from toric geometry

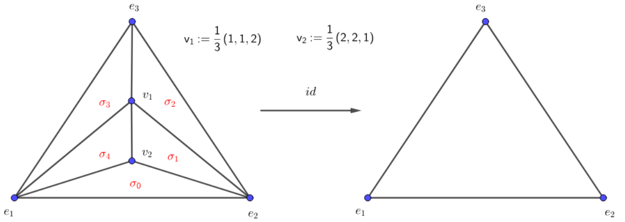
  $\Longrightarrow$ main ingredient: Euler sequence.

- For simplicity we assume $n = 3$ and discuss $\frac{1}{3}(1, 1, 2)$.

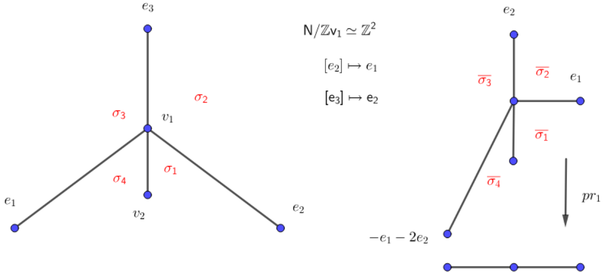
- The singularity $\frac{1}{3}(1,1,2)$ is resolved by two subdivisions of $\sigma = cone(e_1, e_2, e_3)$:



- $\rho\colon \widehat{U} \to U$ induced by inclusion is proper and birational



- To each ray $\mathbb{R}_{\geq 0} e_i$ corresponds a divisor D_i
- The exceptional prime divisors of ρ are the divisors E_i corresponding to $\mathbb{R}_{\geq 0} v_i$
- $E_2 \simeq \mathbb{P}^2$ and $E_1 \simeq \mathbb{F}_2$ (Hirzebruch surface):



A) Rigid manifolds with Kodaira dimension one

- The Euler sequence reads

$$0 \to \mathcal{O}_{\widehat{U}}^{\oplus 2} \to \mathcal{O}_{\widehat{U}}(E_1) \oplus \mathcal{O}_{\widehat{U}}(E_2) \oplus \bigoplus_{i=1}^{3} \mathcal{O}_{\widehat{U}}(D_i) \to \Theta_{\widehat{U}} \to 0.$$

- Push forward using $R^i\rho_*\mathcal{O}_{\widehat{U}} = 0$ ($i \geq 1$) we can show

$$R^1\rho_*\Theta_{\widehat{U}} = 0 \qquad \text{and} \qquad \rho_*\Theta_{\widehat{U}} = \Theta_U.$$

B) Rigid manifolds with non-contractible universal cover

Idea:

Construct an infinitesimally rigid product quotient

$$Y_n = (C_1 \times \ldots \times C_n)/G$$

of general type with finite $\pi_1(Y_n)$.

$\Longrightarrow$ the universal cover of Y_n is projective, whence non-contractible.

Conditions:

We need:

- $g(C_i) \geq 2$ and $C_i \to C_i/G \simeq \mathbb{P}^1$ branched in three points,
- non-free action, otherwise $\pi_1(Y_n)$ is infinite,
- nice singularities i.e.canonical so that $\kappa(X_n) = n$, for any resolution $\rho\colon X_n \to Y_n$,
- a resolution $\rho\colon X_n \to Y_n$ with

$$H^1(\Theta_{X_n}) \simeq H^1(\Theta_{Y_n}) = H^1(\Theta_{C^n})^G = 0.$$

B) Rigid manifolds with non-contractible universal cover

The example:

- Take $C = \{x_0^4 + x_1^4 + x_2^4 = 0\} \subset \mathbb{P}^2$ together with the action

$$\psi\colon \mathbb{Z}_4^2 \to Aut(C), \quad (a,b) \mapsto [(x_0 : x_1 : x_2) \mapsto (\zeta_4^a x_0 : \zeta_4^b x_1 : x_2)]$$

- Define a second action by

$$\psi' := \psi \circ \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

- Y_n is the quotient of C^n by the diagonal $\mathbb{Z}_4^2$-action $\psi \times \psi' \times \ldots \times \psi'$.

Proposition

- The quotient Y_n is infinitesimally rigid, of general type with $3 \cdot 2^{2n-3}$ singularities of type $\frac{1}{2}(1, \ldots, 1)$.
- There is a resolution $\rho\colon X_n \to Y_n$ with $H^1(\Theta_{X_n}) \simeq H^1(\Theta_{Y_n}) = 0$.

Caution:

In the surface case $n = 2$, we have

$$H^1(\Theta_{X_2}) \simeq H^1(\Theta_{Y_2}) \oplus \mathbb{C}^{\# Sing(Y_2)} = \mathbb{C}^{\# Sing(Y_2)}$$

- We still need to show that $\pi_1(X_n) \simeq \pi_1(Y_n)$ is finite.
- Consider $H := \langle (2,0), (0,2) \rangle < \mathbb{Z}_4^2$. Then the natural map

$$C^n/H \to Y_n = C^n/\mathbb{Z}_4^2$$

  is unramified.
- It suffices to show that $Z_n := C^n/H$ has a finite fundamental group.

Remark

Note that $C \to C/H \simeq \mathbb{P}^1$ is branched in 6 points $\implies$ Z_n is not infinitesimally rigid.

B) Rigid manifolds with non-contractible universal cover

Fundamental group of a product quotient:

- Let $X = (C_1 \times \ldots \times C_n)/G$ be a product quotient obtained by a diagonal G action.
- Let $u_i \colon \mathbb{H} \to C_i$ be the universal cover.
- Let $\mathbb{T}_i$ be the group of lifts of the automorphisms of G acting on C_i and $\varphi_i \colon \mathbb{T}_i \to G$ be the natural map.
- Let $\mathbb{G} := \{(x_1, \ldots, x_n) \mid \varphi_1(x_i) = \ldots = \varphi(x_n)\}$ be the fibre product, then it holds

$$\mathbb{H}^n/\mathbb{G} \simeq X.$$

Consequence of Armstrong's Theorem

The fundamental group of $X = (C_1 \times \ldots \times C_n)/G$ is

$$\pi_1(X) \simeq \mathbb{G}/Fix(\mathbb{G}),$$

where $Fix(\mathbb{G})$ is the subgroup generated by the elements with non-empty fixed locus.

Proposition

In the case of Z_n we have $g^2 \in Fix(\mathbb{G})$ for all $g \in \mathbb{G}$.

$\implies$ $\pi_1(Z_n)$ is abelian and therefore finite, because $b_1(Z_n) = 0$ and

$$\pi_1(X) \simeq \pi_1(X)^{ab} \simeq H_1(X, \mathbb{Z}).$$

Thank you for your attention