EILENBERG-MOORE, KLEISLI AND DESCENT FACTORIZATIONS

FERNANDO LUCATELLI NUNES
UTRECHT UNIVERSITY

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- Fundamental aspects

  low/higher dimensional categorical structures

  Universal Algebra

  Monads/Monadicity

  Lawvere Theories

  ...

- Computer Sciences

  Programming Languages

  Software correctness

  Algebraic Effects

  ...

- Fundamental aspects
  - low/higher dimensional categorical structures

$\rightarrow$ <u>Monads / Monadicity</u>

- Fundamental aspects
  - low/higher dimensional categorical structures

$\to$ <u>Monads/Monadicity</u>

* Lucatelli Nunes,
  Semantic Factorization and Descent
  arXiv, 02/2019
  (To appear in TAC)

- Aim of the talk:

→ Sketch one of the consequences in the context of CAT.

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- Aim of the talk:

→ Sketch one of the consequences in the context of CAT.

Categories, functors and natural transformations.

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Semantic Factorization and Descent

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(To appear in TAC)

- $\mathbb{A} \xrightarrow{G} \mathbb{B}$ functor

- $\begin{tikzcd} \mathbb{A} \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\perp(\varepsilon,\eta)"] & & \mathbb{B} \arrow[ll, bend right, "F"'] \end{tikzcd}$ adjoint situation

- $$\begin{tikzcd} \mathbb{A} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp(\varepsilon,\eta)"] & \mathbb{B} \arrow[l, bend right, "F"'] \end{tikzcd}$$ adjoint situation

$\to$ monad $(GF, G\varepsilon F, \eta)$

$\to$ comonad $(FG, F\eta G, \varepsilon)$

(P. J. Huber, 61)

- $$\begin{tikzcd} \mathbb{A} \arrow[rr, phantom, "\perp(\varepsilon,\eta)"] \arrow[rr, bend right, "G"'] & & \mathbb{B} \arrow[ll, bend right, "F"'] \end{tikzcd}$$ adjoint situation

→ monad $(GF, G\varepsilon F, \eta) = T$

→ comonad $(FG; F\eta G, \varepsilon) = V$

- Eilenberg-Moore Factorizations:

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "G"] \arrow[dr, "K_G"'] & & \mathbb{B} \\ & T\text{-}Alg \arrow[ur, "G_T"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} & & \mathbb{B} \arrow[ll, "F"'] \arrow[dl, "K_F"] \\ & V\text{-}CoAlg \arrow[ul, "F_V"] & \end{tikzcd}$$

(Eilenberg and Moore, 1965)

- Examples (Eilenberg – Moore Factorization)

$$\begin{tikzcd} \mathrm{FreeGrp} \arrow[rr] \arrow[dr] & & \mathrm{Set} \\ & \mathrm{Grp} \arrow[ur] & \end{tikzcd}$$

- Examples (Eilenberg – Moore Factorization)

$$\begin{tikzcd}
\mathrm{FreeGrp} \arrow[rr] \arrow[dr] & & \mathrm{Set} \\
& \mathrm{Grp} \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Top} \arrow[rr] \arrow[dr] & & \mathrm{Set} \\
& \mathrm{Set} \arrow[ur, "\text{equiv}"'] &
\end{tikzcd}$$

- Examples (Eilenberg – Moore Factorization)

$$\begin{tikzcd} \mathrm{FreeGrp} \arrow[rr] \arrow[dr] & & \mathrm{Set} \\ & \mathrm{Grp} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Top} \arrow[rr] \arrow[dr] & & \mathrm{Set} \\ & \mathrm{Set} \arrow[ur, "\text{equiv}"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{CompHaus} \arrow[rr] \arrow[dr, "\text{equiv}"'] & & \mathrm{Set} \\ & \mathrm{CompHaus} \arrow[ur] & \end{tikzcd}$$

- Examples (Eilenberg – Moore Factorization)

$$\begin{tikzcd} \mathrm{FreeGrp} \arrow[rr] \arrow[dr] & & \mathrm{Set} \\ & \mathrm{Grp} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Top} \arrow[rr] \arrow[dr] & & \mathrm{Set} \\ & \mathrm{Set} \arrow[ur, "\text{equiv}"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{CompHaus} \arrow[rr] \arrow[dr, "\text{equiv}"'] & & \mathrm{Set} \\ & \mathrm{CompHaus} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{AbGroup} \arrow[rr] \arrow[dr, "\text{equiv}"'] & & \mathrm{Set} \\ & \mathrm{AbGrp} \arrow[ur] & \end{tikzcd}$$

- Examples (Eilenberg–Moore Factorization)

$$\begin{tikzcd} \mathrm{FreeGrp} \arrow[rr] \arrow[dr] & & \mathrm{Set} \\ & \mathrm{Grp} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Top} \arrow[rr] \arrow[dr] & & \mathrm{Set} \\ & \mathrm{Set} \arrow[ur, "\text{equiv}"'] & \end{tikzcd}$$

$\uparrow$ not monadic $\qquad$ $\uparrow$ not monadic

$$\begin{tikzcd} \mathrm{CompHaus} \arrow[rr] \arrow[dr, "\text{equiv}"'] & & \mathrm{Set} \\ & \mathrm{CompHaus} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{AbGroup} \arrow[rr] \arrow[dr, "\text{equiv}"'] & & \mathrm{Set} \\ & \mathrm{AbGrp} \arrow[ur] & \end{tikzcd}$$

$\uparrow$ Monadic $\qquad$ $\uparrow$ Monadic

- Further Examples (Eilenberg - Moore Factorization)

$$\begin{tikzcd} \mathrm{tfAbGroup} \arrow[rr] \arrow[dr] & & \mathrm{Set} \\ & \mathrm{AbGroup} \arrow[ur] & \end{tikzcd}$$

- Further Examples (Eilenberg – Moore Factorization)

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$\to$ not monadic

- Further Examples (Eilenberg – Moore Factorization)

$$\begin{tikzcd}
\mathrm{tfAbGroup} \arrow[rr] \arrow[d, "\text{equiv}"'] \arrow[dr] & & \mathrm{Set} \\
\mathrm{tfAbGroup} \arrow[r] & \mathrm{AbGroup} \arrow[ur] &
\end{tikzcd}$$

→ has a left adjoint

- Further Examples (Eilenberg - Moore Factorization)

$$\begin{tikzcd}
\mathrm{tfAbGroup} \arrow[rr] \arrow[d, "\mathrm{equiv}"'] \arrow[dr] & & \mathrm{Set} \\
\mathrm{tfAbGroup} \arrow[r] & \mathrm{AbGroup} \arrow[ur] &
\end{tikzcd}$$

Last example: $S \to$ small category
$\mathcal{C} \to$ cocomplete and complete

$$\begin{tikzcd}
\mathrm{Fun}(S,\mathcal{C}) \arrow[rr] \arrow[dr, "\mathrm{equiv}"'] & & \mathrm{Fun}(S_0,\mathcal{C}) \\
& \mathrm{Fun}(S,\mathcal{C}) \arrow[ur] &
\end{tikzcd}$$

- Further Examples (Eilenberg - Moore Factorization)

$$
\begin{tikzcd}
\mathrm{tf\,AbGroup} \arrow[rr] \arrow[d, "\mathrm{equiv}"'] \arrow[dr] & & \mathrm{Set} \\
\mathrm{tf\,AbGroup} \arrow[r] & \mathrm{AbGroup} \arrow[ur] &
\end{tikzcd}
$$

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& \mathrm{Fun}(S,\mathcal{C}) \arrow[ur] &
\end{tikzcd}
$$

$\mathrm{Fun}(S,\mathcal{C}) \to$ Algebras and Coalgebras

- Further Examples (Eilenberg - Moore Factorization)

$$\begin{tikzcd}
\mathrm{tfAbGroup} \arrow[rr] \arrow[d, "\mathrm{equiv}"'] \arrow[dr] & & \mathrm{Set} \\
\mathrm{tfAbGroup} \arrow[r] & \mathrm{AbGroup} \arrow[ur] &
\end{tikzcd}$$

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& \mathrm{Fun}(S,\mathbb{C}) \arrow[ur] &
\end{tikzcd}$$

$\to$ Monadic and Comonadic

$\to$ Algebras and Coalgebras

- Getting back to the general setting:

$$\mathbb{A} \xrightarrow{G} \mathbb{B}$$

- General setting:

$\mathbb{A} \xrightarrow{G} \mathbb{B}$ does NOT have Eilenberg-Moore factorization in general.

- General setting:

$\mathbb{A} \xrightarrow{G} \mathbb{B}$ does NOT have Eilenberg-Moore factorization in general!

(We need at least to assume that G has codensity monad)

- General setting:

$$\mathbb{A} \xrightarrow{G} \underbrace{\mathbb{B} \rightrightarrows \mathbb{B} \amalg_G \mathbb{B}}_{\text{cokernel pair of } G}$$

$\mathbb{B} \amalg_G \mathbb{B}$: pushout of G along G in CAT

- General setting:

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "G"] \arrow[d] & \mathbb{B} \arrow[r, shift left] \arrow[r, shift right] & \mathbb{B} \amalg_G \mathbb{B} \\
\mathrm{Eq} \arrow[ur]
\end{tikzcd}$$

$\mathbb{B} \rightrightarrows \mathbb{B} \amalg_G \mathbb{B}$: cokernel pair of G

$\mathbb{B} \amalg_G \mathbb{B}$: pushout of G along G in CAT

Eq $\to$ in CAT

- General setting:

$$\begin{tikzcd} \mathbb{A} \arrow[r, "G"] \arrow[d] & \mathbb{B} \arrow[r, shift left] \arrow[r, shift right] & \mathbb{B} \amalg_G \mathbb{B} \\ \mathrm{Eq} \arrow[ur] & & \end{tikzcd}$$

cokernel pair of G

pushout of G along G in CAT

Eq in CAT

But we won't talk about this factorization!

• General setting: Instead, we take into consideration the 2-dimensional structure of CAT.

$$\mathbb{A} \xrightarrow{G} \underbrace{\mathbb{B} \rightleftarrows \mathbb{B} \,\hat{\uparrow}_G\, \mathbb{B} \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathbb{B} \,\hat{\uparrow}_G\, \mathbb{B} \,\hat{\uparrow}_G\, \mathbb{B}}_{\text{2-dimensional cokernel diagram}}$$

- General setting:

$$\mathbb{A} \xrightarrow{G} \mathbb{B} \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; \underbrace{\mathbb{B} \uparrow_G \mathbb{B}}_{\text{opcomma category of } G \text{ along } G.} \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; \mathbb{B} \uparrow_G \mathbb{B} \uparrow_G \mathbb{B}$$

- General setting:

$$\mathbb{A} \xrightarrow{G} \mathbb{B} \rightleftarrows \underbrace{\mathbb{B} \mathbin{\hat{\uparrow}_G} \mathbb{B}}_{\text{opcomma category of } G \text{ along } G.} \rightrightarrows \mathbb{B} \mathbin{\hat{\uparrow}_G} \mathbb{B} \mathbin{\hat{\uparrow}_G} \mathbb{B}$$

$$\operatorname{colim}\left(\begin{tikzcd} & & \mathbb{B} \\ \mathbb{A} \arrow[urr, "G"] \arrow[dr, "id_{\mathbb{A}} \times i_1"'] & & \\ & \mathbb{A} \times 2 & \\ \mathbb{A} \arrow[ur, "id_{\mathbb{A}} \times i_0"] \arrow[dr, "G"'] & & \\ & \mathbb{B} & \end{tikzcd}\right) \cong \mathbb{B} \mathbin{\hat{\uparrow}_G} \mathbb{B}$$

- General setting:

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "G"] & \mathbb{B} \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbb{B} \uparrow_G \mathbb{B} \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{B} \uparrow_G \mathbb{B} \uparrow_G \mathbb{B}
\end{tikzcd}
$$

colim

$$
\left(
\begin{tikzcd}
 & \mathbb{B} & & \\
\mathbb{A} \arrow[ur, "G"] \arrow[drr, "id_{\mathbb{A}} \times i_1"'] & & & \\
 & & \mathbb{A} \times 3 & & \mathbb{B} \\
\mathbb{A} \arrow[urr, "id_{\mathbb{A}} \times i_0"] \arrow[dr, "G"'] & & & \mathbb{A} \arrow[ul, "id_{\mathbb{A}} \times i_2"] \arrow[ur, "G"'] & \\
 & \mathbb{B} & & &
\end{tikzcd}
\right)
\cong \mathbb{B} \uparrow_G \mathbb{B} \uparrow_G \mathbb{B}
$$

$$
3 = \begin{tikzcd}
2 \\
1 \arrow[u] \\
0 \arrow[u]
\end{tikzcd}
$$

• General setting : 2-dimensional cokernel diagram of G

$$\mathbb{A} \xrightarrow{G} \overbrace{\mathbb{B} \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathbb{B} \uparrow_G \mathbb{B} \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \mathbb{B} \uparrow_G \mathbb{B} \uparrow_G \mathbb{B}}$$

- General setting:

$$\mathbb{A} \xrightarrow{G} \underbrace{\mathbb{B} \rightleftarrows \mathbb{B} \hat{\uparrow}_G \mathbb{B} \rightrightarrows \mathbb{B} \hat{\uparrow}_G \mathbb{B} \hat{\uparrow}_G \mathbb{B}}^{\mathcal{H}_G}$$

$\mathrm{Desc}\,\mathcal{H}_G$ ⟶ 2-dimensional limit / category of descent data

- General setting:

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "G"] \arrow[dr] & \mathbb{B} \arrow[r, shift left] \arrow[r, shift right] & \mathbb{B} \hat{\uparrow}_G \mathbb{B} \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{B} \hat{\uparrow}_G \mathbb{B} \hat{\uparrow}_G \mathbb{B} \\
& \operatorname{Desc}\mathcal{H}_G \arrow[u] & &
\end{tikzcd}
$$

$\mathcal{H}_G$: the diagram $\mathbb{B} \rightleftarrows \mathbb{B} \hat{\uparrow}_G \mathbb{B} \rightrightarrows \mathbb{B} \hat{\uparrow}_G \mathbb{B} \hat{\uparrow}_G \mathbb{B}$

$\operatorname{Desc}\mathcal{H}_G$ → 2-dimensional limit / category of lax descent

→ Universal Properties

- General setting:

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "G"] \arrow[dr] & \mathbb{B} \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbb{B} \uparrow_G \mathbb{B} \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{B} \uparrow_G \mathbb{B} \uparrow_G \mathbb{B} \\
& \mathrm{Desc}\,\mathcal{H}_G \arrow[u] & &
\end{tikzcd}$$

$\mathcal{H}_G$

"Semantic descent factorization" of G

- General setting:

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "G"] \arrow[dr] & & \mathbb{B} \arrow[r, shift left] \arrow[r, shift right] & \mathbb{B} \hat{\top}_G \mathbb{B} \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{B} \hat{\top}_G \mathbb{B} \hat{\top}_G \mathbb{B} \\
& \mathrm{Desc}\,\mathcal{H}_G \arrow[ur] & & &
\end{tikzcd}$$

$\overbrace{\mathbb{B} \rightleftarrows \mathbb{B} \hat{\top}_G \mathbb{B} \Rrightarrow \mathbb{B} \hat{\top}_G \mathbb{B} \hat{\top}_G \mathbb{B}}^{\mathcal{H}_G}$

$\underbrace{\mathbb{A} \to \mathrm{Desc}\,\mathcal{H}_G \to \mathbb{B}}$

"Semantic descent factorization" of G

We should emphasize that EVERY functor G has the semantic descent factorization!

- General setting:

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "G"] \arrow[dr] & \arrow[d, phantom, "\circledast"] & \mathbb{B} \arrow[r, shift left=2] \arrow[r, shift right=2] & \arrow[l] \mathbb{B} \uparrow_G \mathbb{B} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{B} \uparrow_G \mathbb{B} \uparrow_G \mathbb{B} \\
& \mathrm{Desc}\,\mathcal{H}_G \arrow[ur] & & &
\end{tikzcd}$$

$$\underbrace{\mathbb{B} \rightleftarrows \mathbb{B} \uparrow_G \mathbb{B} \rightrightarrows \mathbb{B} \uparrow_G \mathbb{B} \uparrow_G \mathbb{B}}_{\mathcal{H}_G}$$

"Semantic descent factorization" of G

- THEOREM (Lucatelli Nunes, 2019)

G right adjoint $\Longrightarrow$ $\circledast$ coincides w. Eilenberg Moore $\to$

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "G"] \arrow[dr] & & \mathbb{B} \\
& T\text{-Alg} \arrow[ur] &
\end{tikzcd}$$

G left adjoint $\Longrightarrow$ $\circledast$ coincides w.

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "G"] \arrow[dr] & & \mathbb{B} \\
& V\text{-CoAlg} \arrow[ur] &
\end{tikzcd}$$

(*) Semantic descent factorization

THEOREM (Lucatelli Nunes, 2019)

G right adjoint $\Longrightarrow$ (*) coincides w.
$$\begin{tikzcd} \mathbb{A} \arrow[rr, "G"] \arrow[dr] & & \mathbb{B} \\ & T\text{-}\mathrm{Alg} \arrow[ur] & \end{tikzcd}$$

G left adjoint $\Longrightarrow$ (*) coincides w.
$$\begin{tikzcd} \mathbb{A} \arrow[rr, "G"] \arrow[dr] & & \mathbb{B} \\ & V\text{-}\mathrm{CoAlg} \arrow[ur] & \end{tikzcd}$$

Message:

Semantic descent factorization (of any functor) is a (natural) generalization of the Eilenberg-Moore Factorizations

Thank you

for your attention!