Metric Fourier Approximation of Set-Valued Functions of Bounded Variation

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Joint work with Nira Dyn, Elza Farkhi, Alona Mokhov

(Tel Aviv University, Israel)

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Fourier series

$f : \mathbb{R} \to \mathbb{R}$ a 2π-periodic function. The Fourier series of f:

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

$a_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos kt \, dt, \;\; b_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin kt \, dt, \;\; k = (0), 1, 2, \ldots .$

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Partial sums of Fourier seires:

$$\mathscr{S}_n f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) = \frac{1}{\pi} \int_{-\pi}^{\pi} D_n(x-t) f(t) dt,$$

where

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin\left(n + \frac{1}{2}\right) x}{2 \sin\left(\frac{1}{2} x\right)}, \quad x \in \mathbb{R},$$

is the Dirichlet kernel.

Set-valued functions

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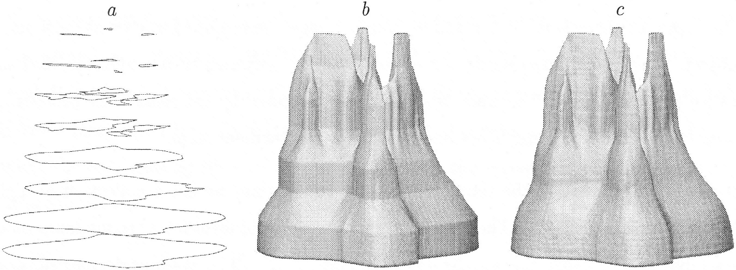

Hausdorff metric

A metric on $\mathbb{R}^d$: $\rho(u,v) = |u - v|$, where $|\cdot|$ is a norm on $\mathbb{R}^d$. (Note that all norms on $\mathbb{R}^d$ are equivalent.)

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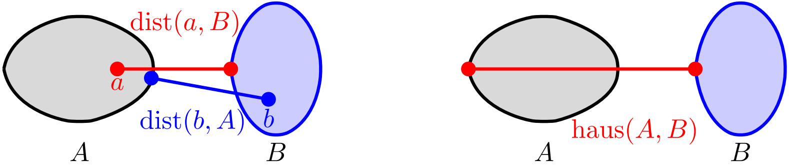

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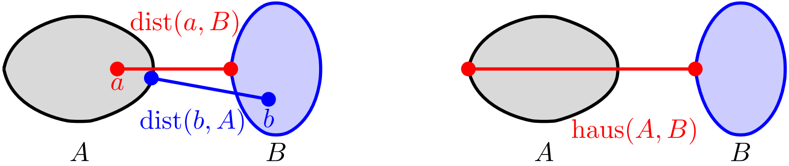

$\mathrm{K}(\mathbb{R}^d)$ is a complete metric space with respect to the Hausdorff metric.

This is how we measure the distance between $F(x)$ and its approximant at a point $x \in [a,b]$.

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Approximation of functions $F : [a, b] \to \mathrm{K}(\mathbb{R}^d)$ with convex compact images: e.g.

- $\to$ R. A. Vitale (1979) — an adoptation of the Bernstein operator,
- $\to$ N. Dyn, E. Farkhi + students (since 80s) — different approaches,
- $\to$ M. Mureșan (2010) — a survey,
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A pioneering work on approximation of SVFs with general, not necessarily convex images:

- → Z. Artstein (1989) — piecewise linear approximation of $F : [a, b] \to \mathrm{K}(\mathbb{R}^d)$ based on metric pairs.

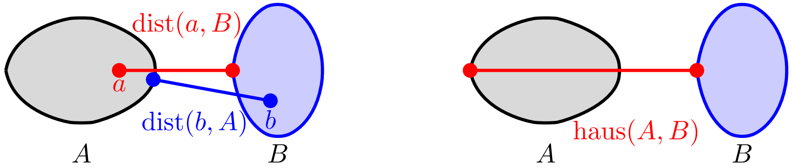
Metric pairs

The set of projections of $a \in \mathbb{R}^d$ on a set $B \in \mathrm{K}(\mathbb{R}^d)$ is

$$\Pi_B(a) = \{b \in B : |a - b| = \mathrm{dist}(a, B)\}.$$

The set of metric pairs of two sets $A, B \in \mathrm{K}(\mathbb{R}^d)$ is

$$\Pi(A, B) = \{(a, b) \in A \times B : a \in \Pi_A(b) \text{ or } b \in \Pi_B(a)\}.$$



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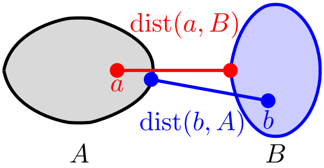

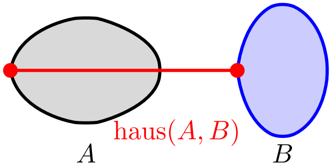

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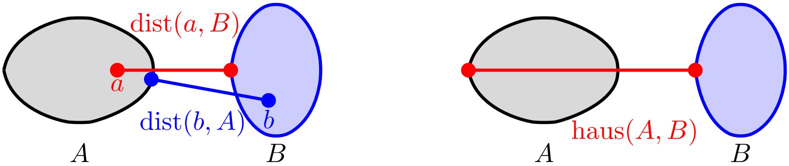

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The idea of Artstein was to create a linear interpolant by connecting metric pairs by pieces of linear functions.

Linear combinations

The standard tool: The Minkowski linear combination of sets $A_1, \dots, A_n \in \mathrm{K}(\mathbb{R}^d)$, $n \geq 1$, is

$$\sum_{i=1}^{n} \lambda_i A_i = \left\{ \sum_{i=1}^{n} \lambda_i a_i : a_i \in A_i \right\}.$$

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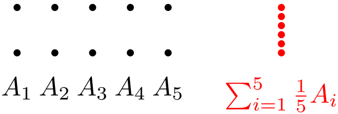

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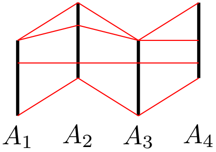
A new tool [Dyn, Farkhi, Mokhov, 2018]:

A metric chain of $A_1, \dots, A_n$ is an n-tuple $(a_1, \dots, a_n)$ such that $(a_i, a_{i+1}) \in \Pi(A_i, A_{i+1})$, $i = 1, \dots, n-1$.



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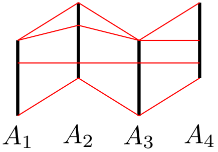

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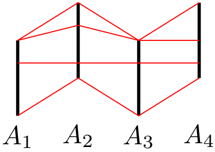

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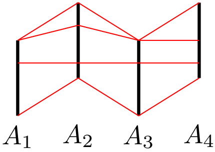

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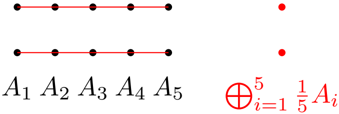

Aumann integral

For a set-valued function $F : [a, b] \to \mathrm{K}(\mathbb{R}^d)$, a single-valued function $s : [a, b] \to \mathbb{R}^d$ such that $s(x) \in F(x)$ for all $x \in [a, b]$ is called a selection of F.

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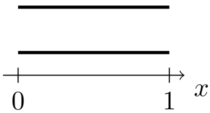

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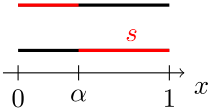

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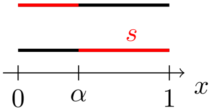

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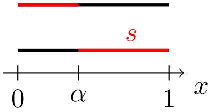

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Metric selections

A new tool [Dyn, Farkhi, Mokhov, 2018]:

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$$c_{\chi,\phi}(x) = \begin{cases} y_i, & x \in [x_i, x_{i+1}), \quad i = 0, \ldots, n-1, \\ y_n, & x = x_n. \end{cases}$$

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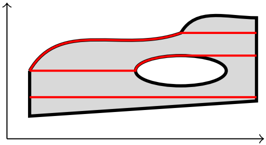


Metric selections

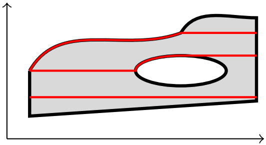
A new tool [Dyn, Farkhi, Mokhov, 2018]:

Given a SVF $F : [a,b] \to \mathrm{K}(\mathbb{R}^d)$, a partition $\chi = \{a = x_0 < x_1 < \cdots < x_n = b\}$, and a corresponding metric chain $\phi = (y_0, \dots, y_n) \in \mathrm{CH}\,(F(x_0), \dots, F(x_n))$, the chain function based on χ and ϕ is

$$c_{\chi,\phi}(x) = \begin{cases} y_i, & x \in [x_i, x_{i+1}), \quad i = 0, \dots, n-1, \\ y_n, & x = x_n. \end{cases}$$

A selection s is called a metric selection, if it is the pointwise limit function of a sequence $\{c_{\chi_k,\phi_k}\}_{k\in\mathbb{N}}$ of chain functions of F, with $\lim_{k\to\infty} |\chi_k| = 0$.

We denote the set of all metric selections of F by $\mathcal{S}(F)$.



A metric selection s is constant in any open interval where the graph of s stays in the interior of $\mathrm{Graph}(F)$.

Metric integral

[Dyn, Farkhi, Mokhov, 2018]: For a SVF $F : [a, b] \to \mathrm{K}(\mathbb{R}^d)$, and for a partition $\chi = \{x_0, \ldots, x_n\}$, the metric Riemann sum of F is defined by

$$^{(\mathcal{M})}S_\chi F = \bigoplus_{i=0}^{n-1} (x_{i+1} - x_i) F(x_i).$$

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Let $\mathcal{F}[a, b]$ be the class of SVFs of bounded variation with closed graphs.

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Result. [Dyn, Farkhi, Mokhov, 2018] Let $F \in \mathcal{F}[a,b]$. Then

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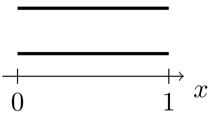

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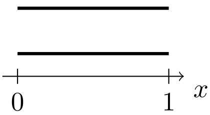

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In [B., Dyn, Farkhi, Mokhov, 2021] we extended the notion of the metric integral to the weighted metric integral of the form

$$(\mathcal{M}_k) \int_a^b k(x)F(x)dx,$$

where $F : [a,b] \to \mathrm{K}(\mathbb{R}^d)$ is a SVF and $k : [a,b] \to \mathbb{R}$ is a weight function.

Metric Fourier series of SVFs

Denote $\partial_{n,x}(t) = D_n(x-t)$.

Definition. [B., Dyn, Farkhi, Mokhov, 2021]) Let $F : [-\pi,\pi] \to \mathrm{K}(\mathbb{R}^d)$. The metric Fourier series of F is the sequence of the set-valued functions $\{\mathscr{S}_n F\}_{n\in\mathbb{N}}$, where $\mathscr{S}_n F$ is a SVF defined by

$$\mathscr{S}_n F(x) = \frac{1}{\pi} (\mathcal{M}_{\partial_{n,x}}) \int_{-\pi}^{\pi} \partial_{n,x}(t) F(t) dt, \quad x \in [-\pi,\pi], \quad n \in \mathbb{N},$$

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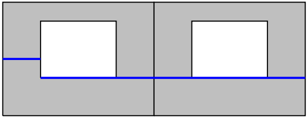
We do not expect metric selections s to be periodic. In fact, even if the set-valued function F itself is periodic, it can have metric selections that are not periodic:



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Theorem 1. [B., Dyn, Farkhi, Mokhov, 2021] Let $F : [-\pi, \pi] \to \mathrm{K}(\mathbb{R}^d)$ be of bounded variation with closed graph. Let F be continuous at $x \in (-\pi, \pi)$. Then

$$\lim_{n\to\infty} \mathrm{haus}\left(\mathscr{S}_n F(x), F(x)\right) = 0.$$

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Theorem 2. [B., Dyn, Farkhi, Mokhov, 2021] Let $F : [-\pi, \pi] \to \mathrm{K}(\mathbb{R}^d)$ be of bounded variation with closed graph and $x \in (-\pi, \pi)$. Then

$$\lim_{n\to\infty} \mathrm{haus}\left(\mathscr{S}_n F(x), A_F(x)\right) = 0,$$

where $A_F(x) = \left\{ \frac{1}{2}\left(s(x+0) + s(x-0)\right) : s \in \mathcal{S}(F) \right\}$.

Thank you for your attention!