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Compactness properties of operator of translation along trajectories in evolution equations

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Let us consider following autonomous wave equation:

$$u_{tt} + \alpha u_t = \Delta u - V(x)u + f(t, x, u), t > 0, x \in \mathbb{R}^N, \tag{1}$$

where $\alpha \geq 0$ is a damping coefficient and V is the so called Kato-Rellich potential, i. e. $V = V_\infty - V_0$, $V_\infty \in L^\infty(\mathbb{R}^N)$, $V_0 \in L^p(\mathbb{R}^N)$, $2 < p < \infty$. In general, the nonlinear forcing term f is continuous and locally Lipschitz (currently we work yet by global Lipschitz condition) with respect to the third variable and T-periodic in time, i.e. $f(t + T, x, u) = f(t, x, u)$. The two cases must be considered separately: the resonant and non-resonant ones.
In the resonant case, that is when the kernel space of the linearization of the $-\Delta + V(x) + f(t, x, \cdot)$ is nontrivial, i. e.

$$\mathcal{N} := \mathrm{Ker}\,(-\Delta + V) \neq \{0\}. \tag{2}$$

We introduce so-called Landesman-Lazer type conditions, which mean that

$$\int_0^T \left(\int_{\{\phi>0\}} \check{f}_+(t,x)\phi(x)\ dx + \int_{\{\phi<0\}} \hat{f}_-(t,x)\phi(x)\ dx \right)\ dt > 0, \qquad (3)$$

for any $\phi \in \mathcal{N} \setminus \{0\}$, where $\check{f}_+(t,x) := \liminf_{s\to+\infty} f(t,x,s)$ and $\hat{f}_-(t,x) := \limsup_{s\to-\infty} f(t,x,s)$ or

$$\int_0^T \left(\int_{\{\phi>0\}} \hat{f}_+(t,x)\phi(x)\ dx + \int_{\{\phi<0\}} \check{f}_-(t,x)\phi(x)\ dx \right)\ dt < 0, \qquad (4)$$

for any $\phi \in \mathcal{N} \setminus \{0\}$, where $\hat{f}_+(t,x) := \limsup_{s\to+\infty} f(t,x,s)$ and $\check{f}_-(t,x) := \liminf_{s\to-\infty} f(t,x,s)$.
It is noteworthy that the conditions of that type can be verified without the explicit knowledge of $\mathcal{N}$ (see Remark 1.3 in [5]).

We are now in point to state our first theorem

Theorem 1

Suppose that we have problem (1) *with assumptions about* $V(x)$ *and* $f(t,x,u)$ *as mentioned above and additionally we request that* f *is bounded by square integrable function, i. e. there exists* $m \in L^2(\mathbb{R}^N)$ *such that for all* $t \in [0,\infty)$, $u \in \mathbb{R}$ *and almost every* $x \in \mathbb{R}^N$

$$|f(t,x,u)| \leq m(x).$$

Moreover, we require that resonance condition (2) *holds as well as one Landesman-Lazer conditions* (3), (4). *Next we assume that*

$$a_\infty := \lim_{R\to+\infty} \operatorname{essinf}_{|x|\geq R} V_\infty(x)$$

is a positive number and $0 \in \sigma_p(-\Delta + V(x))$. *Then there exists* T*-periodic solution of* (1).

The second case concerning situation when there are no resonance, which means that $-\Delta + V(x) + f(t,x,\cdot)$ at zero and infinity has trivial kernels. We shall use the following linearizations of f

$$\lim_{u\to 0} \frac{f(t,x,u)}{u} = \alpha(t,x), \qquad \lim_{|u|\to\infty} \frac{f(t,x,u)}{u} = \omega(t,x), \tag{5}$$

for all $x \in \mathbb{R}^N$ and $t \geq 0$, where $\alpha(t,.),\ \omega(t,.)$ are Kato-Rellich potentials. We work under the assumption that

$$\mathrm{Ker}\,(-\Delta + V - \widehat{\omega}) = \{0\} \quad \text{and} \quad \mathrm{Ker}\,(-\Delta + V - \widehat{\alpha}) = \{0\}, \tag{6}$$

where

$$\widehat{\alpha}(x) := \frac{1}{T}\int_0^T \alpha(t,x)\ dt,\ \ \widehat{\omega}(x) := \frac{1}{T}\int_0^T \omega(t,x)\ dt.$$

Now we formulate theorem concerning nonresonance case.

Theorem 2

Suppose that we have problem (1) *with assumptions about* $V(x)$ *and* $f(t,x,u)$ *as mentioned above. Moreover, we require that nonresonance condition* (6) *holds and linearizations of right-hand-side of* (1) *are topologically different, i. e. numbers of the negative eigenvalues (counted with multiplicities) of* $-\Delta + V - \widehat{\alpha}$ *and* $-\Delta + V - \widehat{\omega}$ *are different modulo 2. Then there exists* T*-periodic solution of* (1).

The hyperbolic equation (1) can be rewritten as the system on space $\mathbf{X} = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$

$$\begin{cases} u_t = v - \delta u \\ v_t = -(-\Delta + V(x))u - (\alpha - \delta)(v - \delta u) + f(t, x, u) \end{cases} \tag{7}$$

with $\delta \geq 0$ (see section 2.1 in [8]). We equip space $\mathbf{X}$ with usual scalar product $(.\,,\,.)_{\mathbf{X}} := (.\,,\,.)_{H^1} + (.\,,\,.)_{L^2}$.

Subsequently we transform formula above into first order differential equation

$$(u_t, v_t) = -(\mathbf{A}_0 + \mathbf{V})(u, v) + \mathbf{F}(t, u, v),\ t > 0, \tag{8}$$

where $\mathbf{A}_0 : D(\mathbf{A}_0) \subset \mathbf{X} \to \mathbf{X}$, is defined by $D(\mathbf{A}_0) = H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and

$$\mathbf{A}_0(u, v) := \Big(\delta u - v, -\Delta u + (\alpha - \delta)v + (\delta^2 - \alpha\delta)u\Big),$$

and $\mathbf{V} : \mathbf{X} \to \mathbf{X}$ by

$$\mathbf{V}(u, v) = (0, V(x)u).$$

The mapping $\mathbf{F} : [0, +\infty) \times \mathbf{X} \to \mathbf{X}$ is given by

$$\mathbf{F}(t, u, v) := (0, F(t, u))$$

with the Nemytskii operator $F : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ given by $[F(t, u)](x) := f(t, x, u(x))$.

It is may be shown, by use of the Lumer-Phillips theorem, that $-(\mathbf{A}_0 + \mathbf{V})$ is the generator of C_0-semigroup of contractions $\left\{\mathbf{S}_{-(\mathbf{A}_0+\mathbf{V})}(t)\right\}_{t\geq 0}$, which enables us to consider mild solutions of (8), i. e we say that function $(u(.), v(.)) : [0, \tau] \to \mathbf{X}$ is a mild solution with initial condition $(\bar{u}, \bar{v})$ if it satisfies following integral formula

$$(u(t), v(t)) = \mathbf{S}_{-(\mathbf{A}_0+\mathbf{V})}(t)(\bar{u}, \bar{v}) + \int_0^t \mathbf{S}_{-(\mathbf{A}_0+\mathbf{V})}(t-s)\mathbf{F}(s, u(s), v(s))\, ds, \quad (9)$$

for any $t \in [0, \tau]$. Consequently, we say that $u : [0, \tau] \to H^1(\mathbb{R}^N)$ is a mild solution of (1) if there exists $v : [0, \tau] \to L^2(\mathbb{R}^N)$ such that (u, v) is a mild solution of (8). In particular, under reasonable assumptions on f, we have the Poincaré operator of translation along trajectories $\mathbf{\Phi}_T : \mathbf{X} \to \mathbf{X}$ associated with (8), defined by

$$\mathbf{\Phi}_T(\bar{u}, \bar{v}) := (u(T), v(T)),$$

where $(u, v) : [0, T] \to \mathbf{X}$ is the mild solution of (8) with the initial condition $(u(0), v(0)) = (\bar{u}, \bar{v})$ (the existence and uniqueness come from standard C_0-semigroup theory, see e.g. [11] or [3]).

Observe that if $(\bar{u}, \bar{v}) \in \mathbf{X}$ is a fixed point of $\mathbf{\Phi}_T$, i.e. $\mathbf{\Phi}_T(\bar{u}, \bar{v}) = (\bar{u}, \bar{v})$, then the mild solution of (8) is T-periodic. Therefore we search for fixed points of $\mathbf{\Phi}_T$.

Let $(E, \|.\|)$ be a Banach space. We say that $G : E \to E$ is *ultimately compact* if for any bounded $D \subset E$ such that

$$D \subset \overline{\text{conv}}(G(D)),$$

then D is relatively compact in E. Observe that any compact map is ultimately compact. We recall that $\text{conv}(D)$ means convex hull of set D and $\overline{\text{conv}}(D)$ means convex closed hull of D (we consider closure in space E). There are possible slightly general definitions, for instance see section 2 in [6]. We are in a position to introduce index theory for ultimately compact sets due to Sadovskii (see Section 3.5.6 in [1]). It is an improvement of index for so-called condensing operators due to Nussbaum and Sadovskii (see Chapter 4 in [10]). One can show that index for ultimately compact operator possesses standard preperties (existence, additivity, homotopy invariance and normalization). Moreover, it is also true, that if G is a compact map, then index for ultimately compact operators coincides with Leray-Schauder index. Further we will denote index of ultimate compact map G with respect to open set $U \subset E$ by $\text{Ind}_{uc}(G, U)$.

According to what was said before we need to prove that Poincaré operator of translation along trajectories is ultimately compact. One of techniques is based on the so-called tail estimates of solutions, for instance estimation of sequences

$$\int_{\{|x|\geq n\}} |u(x,t)|^2 \, dx,$$

where $u : [0,\infty) \to H^1(\mathbb{R}^N)$ is a solution of parabolic equation

$$u_t = \Delta u - V(x)u + f(t,x,u),\ t>0,\ x\in\mathbb{R}^N. \tag{10}$$

The tail estimate method was originated for parabolic equations by Wang in [13]. A tail estimates enabling applications of Conley index to parabolic equation and the associated semiflow on $H^1(\mathbb{R}^N)$ was shown in [12] and adapted in [5]. We shall follow ideas from [6] and [7] for parabolic problems (10) and from [8] for hyperbolic problems.

Assume now that, for all $t \geq 0$, $x \in \mathbb{R}^N$ and $u_1, u_2 \in \mathbb{R}$,

$$|f(t,x,u_1) - f(t,x,u_2)| \leq l(x)|u_1 - u_2|$$

where l is a Rellich-Kato type function. Let u_1, u_2 be two solution of (8). Then $u := u_1 - u_2$ and $v := \delta u + \dot{u}$ satisfy

$$\dot{u} = \Delta u - ((\alpha - \delta)\delta - V)u - (\alpha - \delta)u + F(t,u_1) - F(t,u_2)$$

Furthermore, $\phi : [0,\infty) \to \mathbb{R}$ be smooth function such that $\phi([0,\infty)) \subset [0,1]$, for any $s \in [0,1]$ $\phi(s) = 0$ and for any $s \in [2,\infty)$ $\phi(s) = 1$. Then for any $k \in \mathbb{N}_{\geq 1}$ we put $\phi_k : \mathbb{R}^N \to \mathbb{R}$, $\phi_k(x) := \phi(|x|^2/k^2)$, where $|.|$ stands for norm in $\mathbb{R}^N$. By the regularity theory (see [2])

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(v, v\phi_k)_0 = (\dot{v}, v\phi_k)_0 &= (\Delta u, v\phi_k)_0 + (((\alpha-\delta)\delta - V)u, v\phi_k)_0 + \\ &\quad -(\alpha-\delta)(v, v\phi_k)_0 + (F(t,u_1) - F(t,u_2), v\phi_k)_0 \\ &= I_1 + I_2 + I_3\end{aligned}$$

with

$$I_1 = (\Delta u, v\phi_k)_0, \qquad I_2 = (((\alpha-\delta)\delta - V)u, v\phi_k)_0\,,$$
$$I_3 = -(\alpha-\delta)(v, v\phi_k)_0 + (F(t,u_1) - F(t,u_2), v\phi_k)_0.$$

Clearly

$$I_1 \leq -\frac{1}{2}\frac{d}{dt}(\nabla u, \nabla u\phi_k)_0 - \delta(\nabla u, \nabla u\phi_k)_0 + c_k^{(1)}$$

where $c_k^{(1)} \to 0^+$ as $k \to +\infty$.
Further we have

$$\begin{aligned} I_2 &= (((\alpha-\delta)\delta - a_\infty)u, \dot{u}\phi_k + \delta u\phi_k)_0 + ((a_\infty - V)u, v\phi_k)_0 \\ &= -\frac{1}{2}\frac{d}{dt}((a_\infty - (\alpha-\delta)\delta)u, u\phi_k)_0 - \delta((a_\infty - (\alpha-\delta)\delta)u, u\phi_k)_0 + \\ &\quad + ((a_\infty - V_\infty)u, v\phi_k)_0 + (V_0 u, v\phi_k)_0 \leq \\ &\leq -\frac{1}{2}\frac{d}{dt}((a_\infty - (\alpha-\delta)\delta)u, u\phi_k)_0 - \delta((a_\infty - (\alpha-\delta)\delta)u, u\phi_k)_0 + c_k^{(2)}, \end{aligned}$$

where

$$c_k^{(2)} = (V_0 u, v\phi_k)_0.$$

Clearly, by standard tail estimate techniques

$$c_k^{(2)} \to 0^+ \quad \text{as} \quad k \to +\infty.$$

Finally observe that, that for large k,

$$\begin{aligned} I_3 &= -(\alpha-\delta)(v,v\phi_k)_0 + (F(t,u_1)-F(t,u_2),v\phi_k)_0 \\ &\leq -(\alpha-\delta)(v,v\phi_k)_0 + \mathrm{essup}_{|x|\geq k} l_\infty(x)(|u|,|v|\phi_k)_0 + (l_0|u|,|v|\phi_k)_0 \\ &\leq -(\alpha-\delta)(v,v\phi_k)_0 + \left(\bar{l}_\infty/2\right)\varepsilon(v,v\phi_0) + \left(\varepsilon\bar{l}_\infty/2\right)(u,u\phi_k)_0 + c_k^{(3)} \end{aligned}$$

with

$$\bar{l}_\infty = \lim_{R\to+\infty} \mathrm{essup}_{|x|\geq k} l_\infty(x)$$

$c_k^{(3)} \to 0^+$ as $k \to +\infty$. Hence if we put

$$D(t) := (\nabla u, \nabla u\phi_k)_0 + ((a_\infty - (\alpha-\delta)\delta)u, u\phi_k)_0 + (v, v\phi_k)_0$$

then, by the above esitmates we get

$$\dot{D}(t) \leq -2\delta D(t) - (\alpha-2\delta)(v,v\phi_k)_0 + \bar{l}_\infty/2\varepsilon(v,v\phi_0) + \left(\varepsilon\bar{l}_\infty/2\right)(u,u\phi_k)_0 + c_k$$

with $c_k \to 0^+$ as $k \to +\infty$. If we find $\varepsilon > 0$ and $\delta > 0$ such that

$$-\delta - (\alpha-2\delta) + \frac{L}{2\varepsilon} < 0 \quad \text{and} \quad \frac{\varepsilon L}{2(a_\infty - (\alpha-\delta)\delta)} - \delta < 0 \qquad (11)$$

with $L = \bar{l}_\infty$, then we get $\rho > 0$ such that

$$\dot{D}(t) \leq -\rho D(t) + c_k,$$

Observe that, for any $\delta \in (0, \alpha)$,

$$(\alpha - \delta)\delta \leq (\alpha/2)^2$$

and that if we assume that

$$a_\infty > (\alpha/2)^2$$

and find $\delta > 0$ and $\varepsilon > 0$ such that

$$\delta < \alpha - \frac{L}{2\varepsilon} \quad \text{and} \quad \frac{\varepsilon L}{2(a_\infty - (\alpha/2)^2)} < \delta$$

then these numbers satisfy (11). It can be shown that such numbers exists if

$$L/2 < \alpha^2 (a_\infty - (\alpha/2)^2).$$

Theorem 3

If a_∞, $\bar{l}_\infty$ and α satisfy

$$\bar{l}_\infty < 2\alpha^2 (a_\infty - (\alpha/2)^2),$$

then the Poincaré operator $\mathbf{\Phi}_T : \mathbf{X} \to \mathbf{X}$ is ultimately compact.

Once having the ultimate property, we shall be able to perform fixed point index computations for $\mathbf{\Phi}_T$ using some geometric properties of the equation. The resonant and non-resonant cases are considered separately. We shall follow the ideas from [6] and [7].
In the non-resonant case we shall consider the following family of equations

$$(u_t, v_t) = -(\mathbf{A}_0 + \mathbf{V})(u, v) + \mathbf{F}(t/\varepsilon, u, v),\ t > 0, \tag{12}$$

where $\varepsilon \in (0, 1]$. First we shall show that the non-resonance conditions (6) imply the existence of $r > 0$ and $R > 2r$ such that, for all $\varepsilon \in (0, 1]$, the equation (12) has no nontrivial εT-periodic solution with

$$(u(0), v(0)) \in B_{\mathbf{X}}(0, 2r) \cup (\mathbf{X} \setminus B_{\mathbf{X}}(0, R)).$$

This will enable us to consider the translation along trajectories operator $\mathbf{\Phi}_T^{(\varepsilon)}$ for (12) and ask for the indices $\mathrm{Ind}_{uc}(\mathbf{\Phi}_T^{(\varepsilon)}, B_{\mathbf{X}}(0, r))$ and $\mathrm{Ind}_{uc}(\mathbf{\Phi}_T^{(\varepsilon)}, B_{\mathbf{X}}(0, R))$ where Ind_{uc} denotes fixed point index for ultimately compact mappings. By the homotopy property (note that $\mathbf{\Phi}_T = \mathbf{\Phi}_T^{(1)}$)

$$\mathrm{Ind}_{uc}(\mathbf{\Phi}_T, B_{\mathbf{X}}(0, r)) = \lim_{\varepsilon \to 0^+} \mathrm{Ind}_{uc}(\mathbf{\Phi}_T^{(\varepsilon)}, B_{\mathbf{X}}(0, r))$$

and

$$\mathrm{Ind}_{uc}(\mathbf{\Phi}_T, B_{\mathbf{X}}(0, R)) = \lim_{\varepsilon \to 0^+} \mathrm{Ind}_{uc}(\mathbf{\Phi}_T^{(\varepsilon)}, B_{\mathbf{X}}(0, R)).$$

Then the averaging principle (see e.g. [9] or [4]) will be used to show that, for $U = B_{\mathbf{X}}(0,r)$ and $U = B_{\mathbf{X}}(0,R)$, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0]$, one has

$$\mathrm{Ind}_{uc}(\mathbf{\Phi}_T^{(\varepsilon)}, U) = \mathrm{Ind}_{uc}(\widehat{\mathbf{\Phi}}_{\varepsilon T}, U)$$

where $\widehat{\mathbf{\Phi}}_t$, $t > 0$, are the Poincaré operators associated with

$$u_{tt} + \alpha u_t = \Delta u - V(x)u + \widehat{f}(x,u),\ t > 0,\ x \in \mathbb{R}^N, \tag{13}$$

that is with

$$(u_t, v_t) = -(\mathbf{A}_0 + \mathbf{V})(u,v) + \widehat{\mathbf{F}}(u,v),\ t > 0, \tag{14}$$

where $\widehat{f}(x,u) := \frac{1}{T}\int_0^T f(t,x,u)\,dt$ and $[\widehat{\mathbf{F}}(u,v)](x) := (0, \widehat{f}(x,u(x)))$. Hence, to find the indices

$$\mathrm{Ind}_{uc}(\mathbf{\Phi}_T, B_{\mathbf{X}}(0,r)) \quad \text{and} \quad \mathrm{Ind}_{uc}(\mathbf{\Phi}_T, B_{\mathbf{X}}(0,R))$$

it sufficient to compute the fixed point index of $\widehat{\mathbf{\Phi}}_t$ for small $t > 0$.

To get the fixed point index of $\widehat{\mathbf{\Phi}}_t$ we shall use the linearization method and the spectral properties of operators $-\Delta + V - \widehat{\alpha}$ and $-\Delta + V - \widehat{\omega}$. Namely, we expect that there exists $t_0 > 0$ such that, for all $t \in (0, t_0]$,

$$\mathrm{Ind}_{uc}(\mathbf{\Phi}_T, B_{\mathbf{X}}(0,r)) = (-1)^{m(0)} \quad \text{and} \quad \mathrm{Ind}_{uc}(\mathbf{\Phi}_T, B_{\mathbf{X}}(0,R)) = (-1)^{m(\infty)}$$

where $m(0)$ is the number of negative eigenvalues of $-\Delta + V - \widehat{\alpha}$ (counted with their multiciplities) and $m(\infty)$ is the number of negative eigenvalues of $-\Delta + V - \widehat{\omega}$. Here we use the Weyl spectral theorem to see that the essential spectra of $-\Delta + W_\infty$ and $-\Delta + W_\infty + W_0$ coincide if $W_\infty + W_0$ is a Kato-Rellich potential. Finally, by means of the additivity property of fixed point index, we shall arrive at

$$\mathrm{Ind}_{uc}(\mathbf{\Phi}_T, B_{\mathbf{X}}(0,R) \setminus B_{\mathbf{X}}(0,r)) = (-1)^{m(\infty)} - (-1)^{m(0)},$$

i.e. the formula showing the assertion, i.e. if $m(0) \not\equiv m(\infty)$ mod 2, then $\mathrm{Ind}_{uc}(\mathbf{\Phi}_T, B_{\mathbf{X}}(0,R) \setminus B_{\mathbf{X}}(0,r)) \neq 0$, which implies the existence of T-periodic solution starting from $B_{\mathbf{X}}(0,R) \setminus B_{\mathbf{X}}(0,r)$.

If equation (1) is at resonance, that is $\mathcal{N} = \ker(-\Delta + V) \neq \{0\}$, then we shall consider the following parameterized equation

$$u_{tt} + \alpha u_t = \Delta u - V(x)u + \varepsilon f(t,x,u),\ t > 0,\ x \in \mathbb{R}^N \tag{15}$$

and the associated first order problem

$$(u_t, v_t) = -(\mathbf{A}_0 + \mathbf{V})(u,v) + \varepsilon \mathbf{F}(t,(u,v)),\ t > 0$$

with associated operator of translation $\mathbf{\Phi}_T^{(\varepsilon)}$. Let $P : L^2(\mathbb{R}^N) \to \mathcal{N}$ be the orthogonal projection onto $\mathcal{N}$. Observe that $\mathcal{N} \subset H^1(\mathbb{R}^N)$ and, in view of the Riesz-Schauder theory, $\dim \mathcal{N} < \infty$. Then $L^2(\mathbb{R}^N) = \mathcal{N} \oplus \mathcal{N}^\perp$ and consequently $H^1(\mathbb{R}^N) = \mathcal{N} \oplus \left(\mathcal{N}^\perp \cap H^1(\mathbb{R}^N)\right)$. Subsequently we put $\bar{F}(u) = \frac{1}{T}\int_0^T PF(t,u)\,dt$. We expect that if $U \subset \mathcal{N}$ is an open bounded set such that $\bar{F}(\bar{u}) \neq 0$ for any $\bar{u} \in \partial U$, then for any $r, R > 0$ exists $\varepsilon_0 \in (0,1]$ such that for all $\varepsilon \in (0, \varepsilon_0]$ we have

$$\mathrm{Ind}_{uc}\left(\mathbf{\Phi}_T^{\varepsilon}, (U \oplus B_r) \times B(0,R)\right) = (-1)^{m(\infty)+\dim\mathcal{N}}\mathrm{Deg}_B(\bar{F}, U),$$

where $B_r = \{u \in \mathcal{N}^\perp \cap H^1(\mathbb{R}^N) \mid \|u\|_{H^1} < r\}$, $B(0,R)$ is an open ball in $L^2(\mathbb{R}^N)$ centered in 0 with radius R, $m(\infty)$ is a number of negative eigenvalues (counted with multiplicities) of $-\Delta + V$ and Deg_B denotes topological Brouwer degree.

Next step of our reasoning should be proving resonance version of continuation principle: if there exists $R_0 > 0$ such that $\mathrm{Deg}_B(\bar{F}, B_{\mathcal{N}}(0, R_0)) \neq 0$ $(B_{\mathcal{N}}(0, R_0) = \{u \in \mathcal{N} \mid \|u\|_{H^1} < R_0\})$ and for any $\varepsilon \in (0, 1)$ there are not T-periodic solutions with $\|z(0)\|_{\mathbf{X}} \geq R_0$ of (15), then equation

$$z_t = -(\mathbf{A}_0 + \mathbf{V})z + \mathbf{F}(t, z),\ t > 0 \tag{16}$$

has a T-periodic solution. Hence a crucial point is to show that Brouwer degree associated with $\bar{F}$ is nontrivial. We suppose that Landesman-Lazer conditions allow us to provide such result, and, in consequence, an existence of T-periodic solutions of equations in resonance.

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