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Positive maps and trace polynomials from the symmetric group

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What is this talk about?

"Some multilinear identities and inequalities for matrices"

Inequalities

For all $A, B \geq 0$,

$$\operatorname{tr}(A)\operatorname{tr}(B) - \operatorname{tr}(AB) - \operatorname{tr}(A)B - \operatorname{tr}(B)A + AB + BA \geq 0$$

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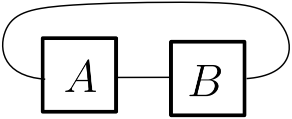
Tensor polynomial identities

For all $x_1, x_2, x_3, x_4 \in M_2$,

$$\sum_{\sigma \in S_4} \epsilon_\sigma x_{\sigma(1)} x_{\sigma(2)} \otimes x_{\sigma(3)} x_{\sigma(4)} = 0$$

Tools

$$\mathrm{tr}(AB) =$$



Tool 1: "permutation to multiplication"

Let (12) be the operator that exchanges two tensor factors.

$$(12) |v\rangle \otimes |w\rangle = |w\rangle \otimes |v\rangle \qquad \forall |v\rangle, |w\rangle \in \mathbb{C}^d$$

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Fact

For all matrices A and B of the same size,

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Recall: partial trace

$$\mathrm{tr}_j : e_1 \otimes \ldots \otimes e_j \otimes \ldots \otimes e_n \mapsto \mathrm{tr}(e_j)\, e_1 \otimes \ldots \otimes e_{j-1} \otimes e_{j+1} \otimes \ldots \otimes e_n$$

alternatively, unique linear operator s.t.

$$\langle A, (\mathbb{1} \otimes B) \rangle = \langle \mathrm{tr}_1(A), B \rangle \quad \forall A \in M_{mn}, B \in M_n$$

Tool 1: "permutation to multiplication" (II)

Let a permutation π exchange k tensor factors.

$$\pi \left|v_1\right\rangle \otimes \left|v_2\right\rangle \otimes \ldots \otimes \left|v_k\right\rangle = \left|v_{\pi^{-1}(1)}\right\rangle \otimes \left|v_{\pi^{-1}(2)}\right\rangle \otimes \ldots \otimes \left|v_{\pi^{-1}(k)}\right\rangle$$

E.g. the permutation $\pi = (143)(2)$ acts on $(\mathbb{C}^d)^{\otimes 4}$ as

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Translate permutation into matrix multiplication.

Permutation to multiplication

For all square matrices $X_1, \ldots, X_k$ of the same size,

$$\operatorname{tr}_{1\ldots k\setminus k}\left[(k\ldots 1)X_1 \otimes X_2 \otimes \ldots \otimes X_k\right] = X_1 X_2 \cdots X_k \, .$$

Tool 2: "positive maps"

Let $\mathcal{P} \in M_{mn}$ with $\mathcal{P} \geq 0$. Then

$$\mathrm{tr}_1[\mathcal{P}(X \otimes \mathbb{1})] \geq 0 \quad \text{for all} \quad X \geq 0$$

Choi-Jamiołkowski isomorphism

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Choi-Jamiołkowski isomorphism

Proof: use “self-duality of the positive cone”:

$$A \geq 0 \quad \Longleftrightarrow \quad \mathrm{tr}[AB] \geq 0 \quad \text{for all} \quad B \geq 0$$

Let’s check. For all $B \geq 0$

$$\mathrm{tr}\{\mathrm{tr}_1[\mathcal{P}(X \otimes \mathbb{1}_n)]B\} = \mathrm{tr}[\mathcal{P}(X \otimes B)] \geq 0$$

(again we used the coordinate-free definition of the partial trace, $\mathrm{tr}[\mathrm{tr}_1(A)B] = \mathrm{tr}[A(\mathbb{1} \otimes B)]$ for all $A \in M_{mn}, B \in M_n$.)

Tool 2: "positive maps" (II)

Multilinear positive maps

For all $\mathcal{P} \geq 0$ and $X_1, \ldots, X_k \geq 0$ of the same size,

$$\mathrm{tr}_{1\ldots k\backslash k}[\mathcal{P}(X_1 \otimes \ldots \otimes X_{k-1} \otimes \mathbb{1})] \geq 0$$

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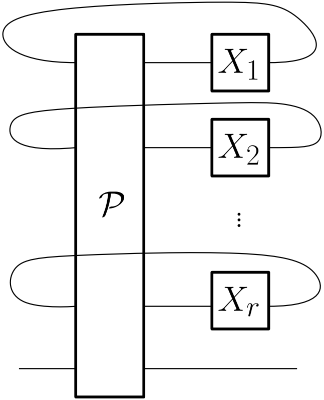

- proof as before, use coordinate-free definition of the partial trace.
- multilinear map from $M_m \times \cdots \times M_m \to M_n$

Put things together

Choose $\mathcal{P} \geq 0$ with $\mathcal{P} = \sum_{\pi \in S_k} a_\pi \pi \in \mathbb{C}S_k$. Then

$$\mathrm{tr}_{1\ldots k \setminus k}[\mathcal{P}(X_1 \otimes \ldots \otimes X_{k-1} \otimes \mathbb{1})] \geq 0$$

. . . is a positive trace polynomial on the positive cone.

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Example

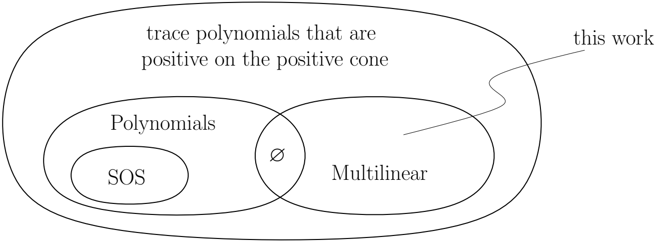
Take $\mathcal{P}_a = (e) - (12) - (23) - (13) + (123) + (132) \geq 0$

Then for all $X, Y \geq 0$

$$\mathrm{tr}(X)\,\mathrm{tr}(Y) - \mathrm{tr}(XY) - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + YX + XY \geq 0\,.$$

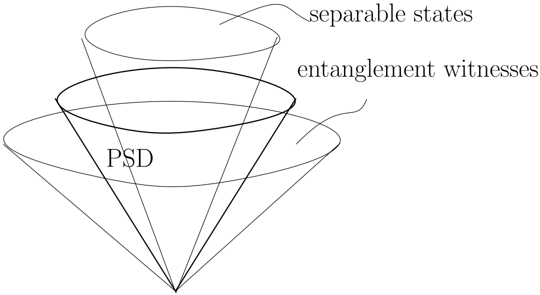
Positive trace polynomials

Which trace polynomials are positive on the positive cone?



(...trace-polynomial: polynomial like expression containing matrix monomials and their traces, e.g. $XYZ + \mathrm{tr}(Y)XZ - 2\,\mathrm{tr}(XZ)\,\mathrm{tr}(Y)\,\mathbb{1}$).

quantum entanglement



Entangled and separable states

A $\varrho \in M_n$ with $\varrho \geq 0$ and $\mathrm{tr}(\varrho) = 1$ is termed a quantum state. Multipartite quantum systems are described by elements in $M_n \otimes \ldots \otimes M_n$.

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Fact

There are quantum states that cannot be written as

$$\sum_i p_i \varrho_i^{(1)} \otimes \ldots \otimes \varrho_i^{(k)} \qquad (\text{“separable state”})$$

where $\varrho_i^{(j)}$ *quantum states and* $pi \geq 0$, $\sum_i p_i = 1$ *probabilities.*

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In other words, if M_n^+ the cone of positive semidefinite matrices. Then

$$\mathrm{conv}\left(\bigotimes^k M_n^+\right) \subsetneq \left(\bigotimes^k M_n\right)^+$$

Entanglement witnesses

An entanglement witness $\mathcal{W} \not\geq 0$ is a matrix, for which

$$\begin{aligned} \mathrm{tr}[\mathcal{W}\varrho] &\geq 0 \quad \text{for all separable } \varrho \\ \mathrm{tr}[\mathcal{W}\varphi] &< 0 \quad \text{for at least one entangled } \varphi \end{aligned}$$

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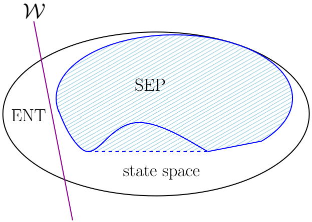

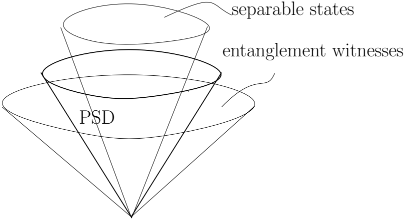

Consider witnesses of the form $\mathcal{W} \in \mathbb{C}S_k$. "Werner state witness"

Positive trace polynomials $\equiv$ Werner state witnesses

Characterization of positive trace polynomials (FH 2021)

Every tight multilinear trace polynomial inequality for the positive cone is in one-to-one correspondence with an optimal Werner state witness through

$$\mathrm{tr}_{1\ldots k \setminus k}[\mathcal{W}(X_1 \otimes \ldots \otimes X_{k-1} \otimes \mathbb{1})]$$

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Example

Take $\mathcal{P}_a = \frac{1}{6}[(e) - (12) - (13) - (23) + (123) + (132)]$.

$$\max_{\varrho \in \mathrm{SEP}} \mathrm{tr}(\mathcal{P}_a \varrho) \leq 1/6$$

Eggeling and Werner, Phys. Rev. A **63**, 042111 (2001)

$\mathcal{W} = \frac{1}{6}\mathbb{1} - \mathcal{P}_a \not\geq 0$ gives

$$\mathrm{tr}(XY) + \mathrm{tr}(X)Y + \mathrm{tr}(Y)X - XY - YX \geq 0 \quad \text{whenever} \quad X, Y \geq 0$$

Tensor polynomial identities

- Recall: the Amitsur-Levitzky Theorem states that on M_2,

$$\sum_{\sigma \in S_4} \epsilon_\sigma X_{\sigma(1)} X_{\sigma(2)} X_{\sigma(3)} X_{\sigma(4)} = 0$$

  is a polynomial identity of minimal degree.

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- New: tensor polynomial identity for M_2

$$\sum_{\sigma \in S_4} \epsilon_\sigma X_{\sigma(1)} X_{\sigma(2)} \otimes X_{\sigma(3)} X_{\sigma(4)} = 0$$

Tensor polynomial identities II

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"All multilinear tensor polynomial identities are consequences of the Cayley-Hamilton theorem."

...more precisely: the expression

$$\mathrm{tr}_{1\ldots k}(\alpha X_1 \otimes \ldots \otimes X_k \otimes \mathbb{1}^{\otimes m})$$

is a multilinear trace polynomial identity on M_n if and only if $\alpha \in \mathbb{C}S_{k+m}$ belongs to the ideal that corresponds to partitions $\lambda \vdash (k+m)$ with more than n parts, whose permutations contain exactly m cycles, each of which move exactly one of the last m positions.

As in $\mathrm{tr}_{1234}\left[(215)(436) X_1 \otimes X_2 \otimes X_3 \otimes X_4 \otimes \mathbb{1} \otimes \mathbb{1}\right] = X_1 X_2 \otimes X_3 X_4$

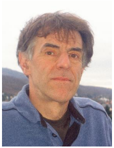
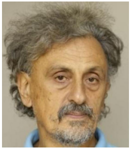
Further work

- Every immanant inequality can be lifted to a trace polynomial inequality for the positive cone.

FH and Hans Massen, *Matrix forms of immanant inequalities*, arXiv:2103.04317

- Complete characterization of all multilinear alternating tensor polynomials in n^2 variables in terms of Young diagrams.

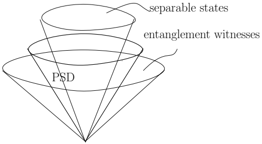
FH and Claudio Procesi, *Tensor polynomial identities*, arXiv:2011.04362, accepted at Israel J. Math.

Summary

- Tools: map permutations to matrix products & positive maps.
- Entanglement witnesses characterize all trace polynomials that are positive on the positive cone.
- All tensor polynomial identities are consequences of the Cayley-Hamilton theorem.

FH, *Positive maps and trace polynomials from the symmetric group*,
J. Math. Phys. **62**, 022203 (2021); arXiv:2002.12887.



…thank you for your attention!

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