The topological conjugacy criterion for surface Morse-Smale flows with a finite number of moduli

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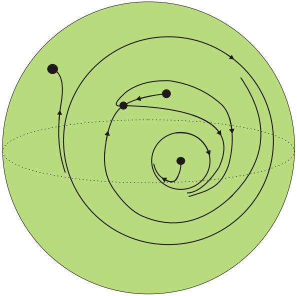
Topological conjugacy and equivalence

- Two flows $f^t, f'^t \colon M \to M$ on a manifold M are called *topologically equivalent* if there exists a homeomorphism $h \colon M \to M$ sending trajectories of f^t into trajectories of f'^t preserving orientations of the trajectories.
- Two flows are called *topologically conjugate* if $h \circ f^t = f'^t \circ h$, it means that h sends trajectories into trajectories preserving not only directions but in addition the time of moving.
- To find an invariant showing the class of topological equivalence or topological conjugacy of flows in some class means to get a *topological classification* for the class.

The Morse-Smale flows

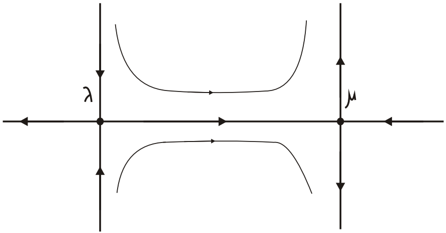
A flow on a surface is called Morse-Smale if its non-wandering set consists of a finite number of hyperbolic fixed points and finite number of hyperbolic limit cycles, besides, there is no trajectories connecting saddle points.



The most important topological invariants for Morse-Smale flows are the *Leontovich-Maier's scheme* $[4, 5]$, the *Peixoto's directed graph* $[6]$ and the *Oshemkov-Sharko's molecule* $[7]$.

The moduli of stability

- A separatrix connecting saddle points gives infinitely many conjugacy classes in one equivalence class, described by a modulus $\frac{\lambda}{\mu}$ called the *modulus of stability* (Palis, 1978).



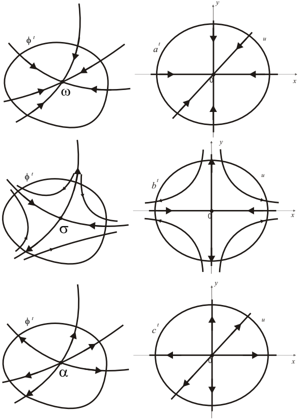
- For surface gradient-like flows classes of topological equivalence and topological conjugacy on surfaces coincide (Kruglov, [2]).
- Any limit cycle obviously generates a *modulus* equal to its period

The problems solved in the work

- The criterion of the moduli finiteness for the surface Morse-Smale flows;
- Topological classification in sense of conjugacy for surface Morse-Smale flows with a finite number of moduli.

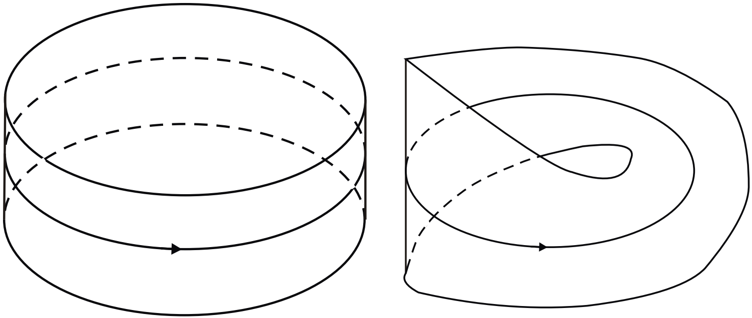
Fixed points

The hyperbolicity of fixed points leads to the following types of fixed points: a sink, a saddle and a source. A flow near a fixed point is topologically conjugate with a linear flow with a sink, saddle or source respectively (Palis, de Melo $[9]$, Robinson $[10]$, Kruglov [2]).



Limit cycles

The hyperbolicity of limit cycles leads to the fact that limit cycles may be only stable or unstable. The neighbourhood of a limit cycle is an annulus or a Möbius band.



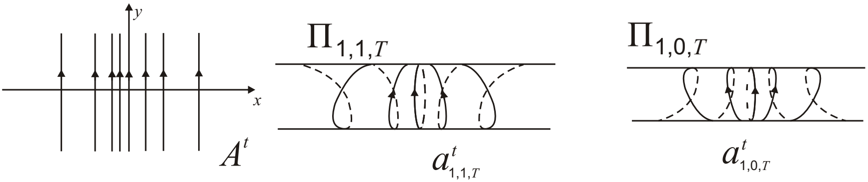
Linearisation near limit cycles

We define a flow $A^t : \mathbb{R}^2 \to \mathbb{R}^2$ as $A^t(x, y) = (x, y + t)$. For $\mu \in \{-1, 1\}$, $\lambda \in \{0, 1\}$ and $T > 0$ let us consider a homeomorphism $g_{\mu,\lambda,T} : \mathbb{R}^2 \to \mathbb{R}^2$ given by the formula

$$g_{\mu,\lambda,T}(x, y) = (\mu \cdot 2^{(-1)^{\lambda+1}} x, y - T)$$

and the group $G_{\mu,\lambda,T} = \{g^n_{\mu,\lambda,T}, n \in \mathbb{Z}\}$. Denote by $\Pi_{\mu,\lambda,T}$ a space orbit of the action of the group $G_{\mu,\lambda,T}$ on $\mathbb{R}^2$ and by $q_{\mu,\lambda,T} : \mathbb{R}^2 \to \Pi_{\mu,\lambda,T}$ the natural projection. Then $\Pi_{\mu,\lambda,T}$ is a cylinder for $\mu = 1$ and a Möbius band for $\mu = -1$; the flow A^t induces by $q_{\mu,\lambda,T}$ the flow $a^t_{\mu,\lambda,T}$ on $\Pi_{\mu,\lambda,T}$ with unique stable limit cycle $c_{\mu,\lambda,T} = q_{\mu,\lambda,T}(Oy)$ of the period T for $\lambda = 0$, and the flow $a^t_{\mu,\lambda,T}$ on $\Pi_{\mu,\lambda,T}$ with unique unstable limit cycle $c_{\mu,\lambda,T} = q_{\mu,\lambda,T}(Oy)$ of the period T for $\lambda = 1$.

Linearisation near limit cycles

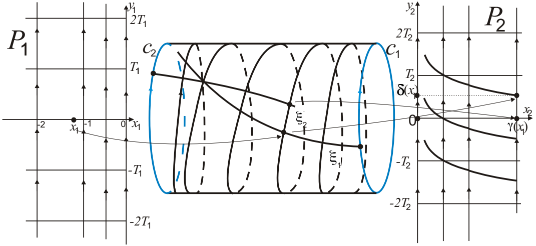


Proposition (Irwin [1])

For every hyperbolic limit cycle c_i of a flow $\phi^t : S \to S$ there are numbers $\mu_i \in \{-1, 1\}$, $\lambda_i \in \{0, 1\}$, $T_i > 0$ and a neighbourhood U_i such that $\phi^t|_{U_i}$ is topologically conjugate to the flow $a^t_{\mu_i,\lambda_i,T_i}$.

The limit cycle c_i is called *a stable, an unstable* for $\lambda_i = 0, 1$ respectively.

The unique foliation near limit cycle



Let $K_i = W^u_{\Omega_i}$ for an unstable cycle Ω_i and $K_i = W^s_{\Omega_i}$ for a stable cycle Ω_i, respectively.

Proposition (Kruglov, Pochinka, Talanova, [3])

There is a unique one-dimensional foliation Ξ_i in K_i whose leaves ξ_i are cross-sections for trajectories of flow $\phi^t|_{K_i}$ and

$$\phi^{T_i}(z) \in \xi_i,\ \phi^t(z) \notin \xi_i \text{ for } 0 < t < T_i,\ \text{if } z \in \xi_i.$$

The moduli finiteness condition

Recall that a modulus of topological conjugacy is an analytical parameter describing infinite many conjugacy classes in the equivalence class.
The first main result of the report is the following.

Theorem

A Morse-Smale surface flow has a finite number of moduli iff it has no a trajectory going from one limit cycle to another.

Let G be the class of Morse-Smale flow with a finite number of moduli, and let $\phi^t \in G$.

Cutting set and cutting circles. An elementary region

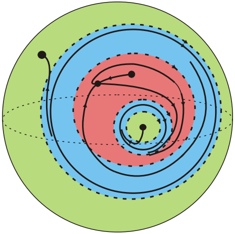
Let $\mathcal{R} = \bigcup\limits_{\mathfrak{c} \in \Omega^3_{\phi^t}} R_{\mathfrak{c}}$ be the union of the boundary circles of cycles'es neighbourhoods. We call $\mathcal{R}$ a *cutting set* and the connected components of $\mathcal{R}$ *cutting circles*. Let $\hat{S} = S \setminus \mathcal{R}$. We call *an elementary region* a connected component of the set $\hat{S}$. The elementary regions, obviously, can be of the following pairwise disjoint types with respect to information about basic sets of ϕ^t in the regions:

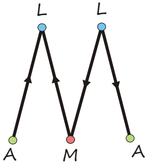
1) a region of the type $\mathcal{L}$ contains exactly one limit cycle;

2) a region of the type $\mathcal{A}$ contains exactly one source or exactly one sink;

3) a region of the type $\mathcal{M}$ contains at least one saddle point;

The directed graph of a flow





Definition

A directed graph Υ_{ϕ^t} is said to be a *graph of the flow* $\phi^t \in G$ if

(1) the vertices of Υ_{ϕ^t} bijectively correspond to the elementary regions of ϕ^t;

(2) every directed edge of Υ_{ϕ^t}, which joins a vertex a with a vertex b, corresponds to the cutting circle R, which is a common boundary of the regions A and B corresponding to a and b, such that any trajectory of ϕ^t passing R goes from A to B by increasing the time.

Properties of the directed graph

We will call a $\mathcal{L}$-, $\mathcal{A}$-, or $\mathcal{M}$*-vertex* a vertex of Υ_{ϕ^t}, which corresponds to a $\mathcal{L}$-, $\mathcal{A}$-, or $\mathcal{M}$-region accordingly.

Proposition

Let Υ_{ϕ^t} be the directed graph of a flow $\phi^t \in G$, then:

1) every $\mathcal{M}$-vertex can be connected only with $\mathcal{L}$-vertices, furthermore, with every vertex by a single edge;

2) every $\mathcal{A}$-vertex can be connected only with a $\mathcal{L}$-vertex, furthermore, by a single edge;

3) every $\mathcal{L}$-vertex has degree (the number of incident edges) 1 or 2, and if its degree is 2, then both edges either enter the vertex or exit.

Equipping of the graph Υ_{ϕ^t}

The flows in $\mathcal{A}$-regions can belong to only the two conjugacy classes: a source pool and a sink pool, which we can distinguish by directions of edges incident to $\mathcal{A}$-vertices.

The flows in $\mathcal{L}$-regions can belong to only the four equivalence classes:
– an annulus with a stable limit cycle;
– an annulus with an unstable one;
– the Möbius band with a stable one;
– the Möbius band with an unstable one.
But every equivalence class consists of infinitely many conjugacy classes depending on a period of limit cycles. So, let us equip each $\mathcal{L}$-vertex with a *cycle modulus*, i.e. the period.

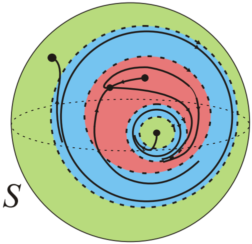
Equipping of $\mathcal{M}$-vertex. Constructing a surface M and a gradient-like flow on it

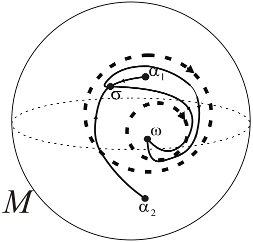
Consider an $\mathcal{M}$-region. It can be

- a 2-manifold with a boundary (with “holes”);
- a closed surface;

Attach a union D of 2-disk to each boundary component of $\mathcal{M}$ to get a closed surface M.

Let $f^t\colon M \to M$ be te flow such that $f^t|_{\mathcal{M}} = \phi^t|_{\mathcal{M}}$ and that Ω_{f^t} has exactly one sink or one source in each connected component of D.





Equipping of $\mathcal{M}$-vertex. A cell

Let $\Omega^0_{f^t}$, $\Omega^1_{f^t}$, $\Omega^2_{f^t}$ be the sets of all sources, saddle points and sinks of f^t accordingly. By the definition of the region $\mathcal{M}$ the flow f^t has at least one saddle point. Let

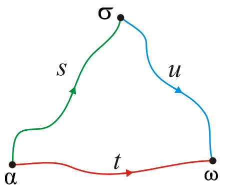
$$\tilde{M} = M \setminus (\Omega^0_{f^t} \cup W^s_{\Omega^1_{f^t}} \cup W^u_{\Omega^1_{f^t}} \cup \Omega^2_{f^t}).$$

A connected component of $\tilde{M}$ is called *a cell*.

Proposition (Peixoto [7])

Every cell J of the flow f^t contains a single sink ω and a single source α in its boundary, and the whole cell is the union of trajectories going from α to ω.

Equipping of $\mathcal{M}$-vertex. A triangle region



Let us choose a t-curve in each cell J which is some usual trajectory in J. Let us call an u-*curve* an unstable saddle separatrix with a sink in its closure, an s-*curve* a stable saddle separatrix with a source in its closure. We will call a *triangle region* Δ the connecting component of $\bar{M}$.

Proposition (Oshemkov-Sharko [6])

Every triangle region Δ is homeomorphic to an open disk and its boundary consists of an unique t-curve, an unique u-curve and an unique s-curve.

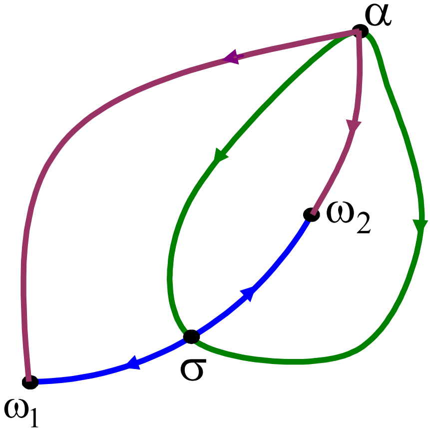
The three-colour graph for a flow

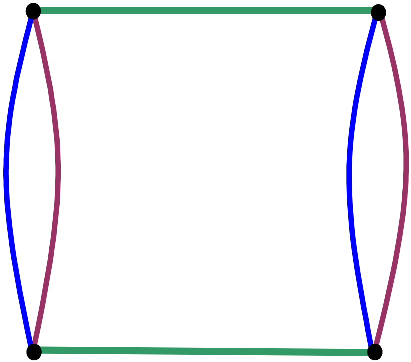
We say that a three-colour graph $\Gamma_{\mathcal{M}}$ corresponds to f^t if:
1) the vertices of $\Gamma_{\mathcal{M}}$ bijectively correspond to the triangle regions of Δ_{f^t};
2) two vertices of $\Gamma_{\mathcal{M}}$ are incident to an edge of colour s, t or u if the polygonal regions corresponding to these vertices has a common s-, t- or u–curve; that establishes an one-to-one correspondence between the edges of $\Gamma_{\mathcal{M}}$ and the colour curves;

Definition

We say that the graph $\Gamma_{\mathcal{M}}$ is the three-colour graph of the flow f^t corresponding to $\phi^t|_{\mathcal{M}}$.

A flow and its three-colour graph





Equipment of some directed edges

Let us denote by π_{f^t} the correspondence described above between elements of f^t and $\Gamma_{\mathcal{M}}$.
Let ut-, st- and su cycles be the cycles of $\Gamma_{\mathcal{M}}$ consisting only of the edges of corresponding colours.

Proposition

The projection π_{f^t} gives an one-to-one correspondence between the sets $\Omega^0_{f^t}$, $\Omega^1_{f^t}$, $\Omega^2_{f^t}$ and the sets of tu-cycles, su-cycles of the length 4, and st-cycles respectively.

By our construction $M = \mathcal{M} \cup D$ each connected component of D contains one sink ω (source α) corresponding to $R_{\mathfrak{c}}$ for $\mathfrak{c}$ of ϕ^t, which corresponds to an $(\mathcal{M}, \mathcal{L})$-edge ($(\mathcal{L}, \mathcal{M})$-edge) of Υ_{ϕ^t}. Thus we induce an orientation from $R_{\mathfrak{c}}$ to the cycle.

The equipped graph

Definition

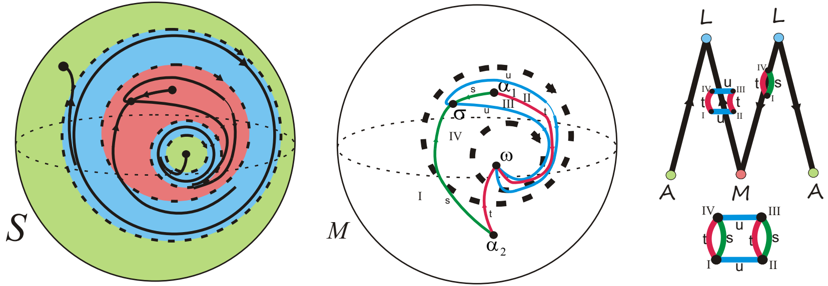
Let Υ_{ϕ^t} be the directed graph of a flow $\phi^t \in G$. We will say that Υ_{ϕ^t} is the *equipped graph* of ϕ^t and denote it by $\Upsilon^*_{\phi^t}$ if:

(1) every $\mathcal{M}$-vertex is equipped with a four-colour graph $\Gamma_{\mathcal{M}}$ corresponding to the flow f^t constructed before;

(2) every edge $(\mathcal{M}, \mathcal{L})$ $((\mathcal{L}, \mathcal{M}))$ is equipped with an oriented tu-cycle (st-cycle) $\tau_{\mathcal{M},\mathcal{L}}$ $(\tau_{\mathcal{L},\mathcal{M}})$ of $\Gamma_{\mathcal{M}}$ corresponding to the limit cycle $\mathfrak{c}$ of $\mathcal{L}$ and oriented consistently with $R_{\mathfrak{c}}$.

(3) every $\mathcal{L}$-vertex is equipped with the cycle modulus $T_{\mathfrak{c}}$.

An example of the equipped graph construction



The classification result

Definition

Equipped graphs $\Upsilon^*_{\phi^t}$ and $\Upsilon^*_{\phi'^t}$ are said to be *isomorphic* if there is an one-to one correspondence ξ between all edges and vertices of $\Upsilon^*_{\phi^t}$ and all edges and vertices of $\Upsilon^*_{\phi'^t}$ preserving their equipments in the following way:

(1) the cycle moduli of vertices $\mathcal{L}$ and $\xi(\mathcal{L})$ are equal;

(2) for vertices $\mathcal{M}$ and $\xi(\mathcal{M})$, there is an isomorphism $\psi_{\mathcal{M}}$ of the three-colour graphs $\Gamma_{\mathcal{M}}, \Gamma_{\xi(\mathcal{M})}$ such that $\psi_{\mathcal{M}}(\tau_{\mathcal{M},\mathcal{L}}) = \tau_{\xi(\mathcal{M}),\xi(\mathcal{L})}$ and the orientations of $\psi_{\mathcal{M}}(\tau_{\mathcal{M},\mathcal{L}})$ and $\tau_{\xi(\mathcal{M}),\xi(\mathcal{L})}$ coincide (similarly for $\tau_{\mathcal{L},\mathcal{M}}$).

Theorem

Flows $\phi^t, \phi'^t \in G$ are topologically conjugate if and only if the equipped graphs $\Upsilon^*_{\phi^t}$ and $\Upsilon^*_{\phi'^t}$ are isomorphic.

References

[1] M. C. Irwin. A classification of elementary cycles. Topology, 9:35-47, 1970.
[2] V. E. Kruglov. Topological conjugacy of gradient-like flows on surfaces. Dinamicheskie sistemy, 8(36)(1):15-21, 2018.
[3] V. Kruglov, O. Pochinka, G. Talanova. On functional moduli of surface flows. Proceedings of the International Geometry Center, 13(1): 49-60, 2020.
[4] E. A. Leontovich, A. G. Mayer. On trajectories determining qualitative structure of sphere partition into trajectories. Doklady Akademii nauk SSSR, 14(5):251-257, 1937 (in Russian).
[5] E. A. Leontovich, A. G. Mayer. On a scheme determining the topological structure of the separation of trajectories. Dokl. Akad. Nauk SSSR (N.S.), 103:557-560, 1955 (in Russian).

References

[6] M. M. Peixoto. On the classification of flows on 2-manifolds. In Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), pages 389-419, 1973.
[7] A. A. Oshemkov, V. V. Sharko. On the classification of Morse-Smale flows on two-dimensional manifolds. Mat. Sb., 189(8):93-140, 1998.
[8] J. Palis. A differentiable invariant of topological conjugacies and moduli of stability. Asterisque, 51:335-346, 1978.
[9] J. Palis J., W. de Melo. (1982). Geometric theory of dynamical systems. New York, Heidelberg, Berlin, Springer-Verlag.
[10] C. Robinson (1995). Dynamical systems: stability, symbolic dynamics, and chaos. CRC Press, Boca Raton, Ann Arbor, London, Tokyo.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