On the Asymmetrizing Cost and Density of Graphs

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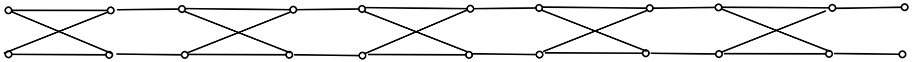
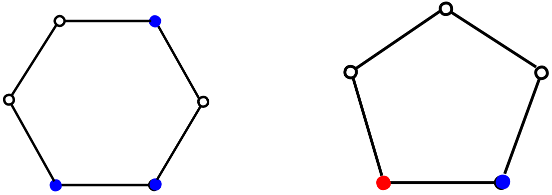
This talk is mainly about automorphism breaking of vertex transitive cubic graphs .

It is joint work with
Gundelinde Wiegel, Thomas Lachmann, and Thomas Tucker



Automorphism breaking

Let me begin with two examples:



In both graphs the only automorphism that preserve the coloring is the identity. Such colorings are called distinguishing or asymmetrizing.

We wish

• to use as few colors as possible,

• and to minimize the number of vertices that are not white. By abuse of language we call them the colored vertices.

Why subcubic graphs, i.e. graphs with maximal valence 3?

A lot is known about automorphism breaking of subcubic graphs G,
but not much about the minimum number of colors needed.

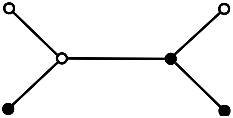
The automorphisms of finite or infinite connected graphs subcubic G can be broken by 2-colorings, unless G is

one of six exceptional graphs

or a member of one of five well known classes of finite graphs[1].

In many cases one has to color almost one half, or even exactly one half of the vertices to break all automorphisms.

The simplest example for this extreme case is an edge, but it can easily be extended to graphs of arbitrary size.



[1] Svenja Hüning, Wilfried Imrich, Judith Kloas, Hannah Schreiber and Thomas Tucker, Distinguishing graphs of maximum valence 3, Electron. J. Comb. 26 (2019).

Vertex transitive graphs

By contrast, if the graphs are vertex transitive, then one can "usually" break the automorphisms by coloring at most five vertices black.

If the graphs are infinite five black vertices may suffice, otheriwse infinitely many are needed.

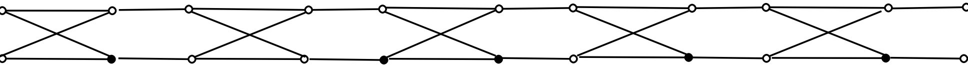
In that case the density of the set of colored vertices is "usually" zero. But it can also be positive.

We will focus on the unusual:

• on finite, connected v.t. cubic graphs, for which
more than five colored vertices are needed to break their automorphisms,

• and on infinite, connected v.t. cubic graphs G for which
one has to color a set $S \subset V(G)$ of positive density to break $\mathrm{Aut}(G)$.

Distinguishing the chain of quadrangles



Questions

Can we distinguish the chain by coloring only finitely many vertices?

Can we reduce the percentage of colored vertices?

Are there graphs where other positive percentages of vertices have to be colored to break all automorphisms?

Does the size of the automorphism group play a role?

The distinguishing number and the distinguishing cost

• In 1996 Albertson and Collins introduced the term distinguishing number $D(G)$ for the minimum number of colors needed to distinguish a graph G by a vertex coloring.

$D(G)$ is 1 for asymmetric graphs and 2 for almost all other finite graphs.

• We primarily consider graphs with distinguishing number 2.
In this case each two colors, black and white, suffice.

For the smallest number of black vertices Boutin (2008) introduced the term 2-distinguishing cost $\rho(G)$.

For most asymmetric graphs $\rho(G) = 1$.

Asymmetrizing vs. distinguishing

Asymmetrizing was elusive until 1996, but as soon as the term distinguishing number was introduced. it became popular as distinguishing.

Albertson and Collins' paper spawned hundreds of publications.

Most, but not all in the context of graphs, but, also to groups or maps.

The distinguishing density – definition

First we define the density $\delta_v(S)$ of a set S of vertices at a vertex v as

$$\delta_v(S) = \limsup_{n\to\infty} \frac{\mid B(v,n)\cap S \mid}{\mid B(v,n) \mid}$$

If $\delta_v(S)$ exists for all vertices, then the density of S is

$$\delta(S) = \sup_{v\in V(G)} \{\delta_v(S) v \in V(G)\}.$$

The distinguishing density $\delta(G)$ of G is then[2]

$$\inf_{S\subseteq V(G),\ \text{distinguishing}} \delta(S)$$

[2] S.M. Smith, F. Lehner, W.I., 2020.

When is the density well defined?

For cubic graphs density zero is always well defined.

Positive density is only interesting for v.t. graphs.

Here a conditon slightly stronger than polynomial growth suffices.

Properties of the infinite chain of quadrangles K

$\mathrm{Aut}(K)$ is uncountable. By the following theorem $\rho(K)$ is infinite.

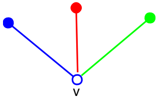
Theorem [D. Boutin, W.I. 2017] Let G be a connected, locally finite graph with infinite automorphism group. Then $\rho(G)$ is finite if and only if $\mathrm{Aut}(G)$ is countable.

Arc orbits of v.t. cubic graphs

The orbit of an arc vw is

$$O(vw) = \{xy \mid x = \alpha(v), y = \alpha(w), \alpha \in \mathrm{Aut}(G)\}.$$

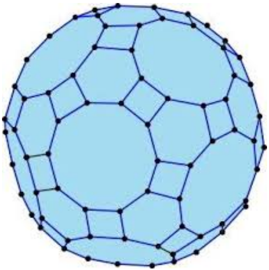
By vertex-transitivity every vertex has to be incident to at least one arc from every arc orbit, hence the number of arc orbits in a vertex-transitive cubic graph is 1,2 or 3.



Three arc orbits

Consider the truncated icosidodecahedron, already described by Johannes Kepler.

Each vertex is in a square, a hexagon, and a decagon.



Clearly it can be distinguished by coloring a single vertex black.

In general one can say: if a connected graph G is v.t. cubic with three arc orbits, then $\rho(G) = 1$.

One arc orbit

Such graphs are also called arc-transitive.

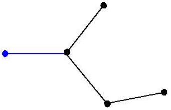
With results from Tutte and Djokovic about cubic arc-transitive graphs one can prove:

Theorem *Let G be an arc-transitive cubic graph different from K_4, $K_{3,3}$, the cube and the Petersen graph.*

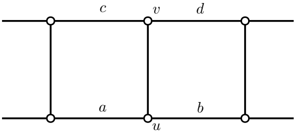
If G has finite girth, then $\rho(G) \leq 5$.

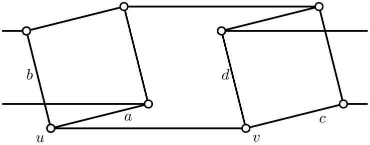
Otherwise $G = T_3$, $\rho(T_3) = \infty$, and $\delta(T_3) = 0$.

Two arc orbits



There are two fundamentally different types of v.t. cubic graphs with two arc-orbits. Compare the edges uv in the two figures.





If one interchanges the edges a and b in the ladder, then c, d also have to be interchanged. This is not the case in the figure on the right.

We call these graphs rigidly, resp. flexibly connected.

Theorem *Let G be a connected vertex-transitive cubic graph with two arc orbits that is rigidly connected. Then $\rho(G) \leq 3$.*

Flexibly connected graphs

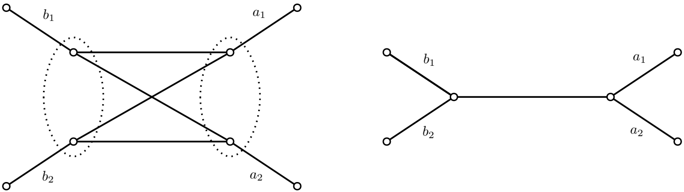
The case when the non-matching edges form triangles can easily be treated:

If we contract the triangles to single vertices, then we obtain an arc transitive cubic graph.

One can then show, with the result about arc-transitive cubic graphs, that $\rho(G) \leq 5$, or $\delta(G) = 0$.

Two arc orbits, where one arc-orbit consists of quadrangles, flexible connection

We define a folding operation for the quadrangles



Folding of quadrangles

With this operation the chain of quadrangles is folded to a chain of single and double edges

.

Defolding

It is easily seen that the process of defolding is unique.

Furthermore, distinguishing colorings of a graph obtained by folding can be extended to the original graph.

One can thus characterize the cost and density of graphs that can be folded to a finite ring or infinite chain of single and double edges.

Graphs that can be folded to a ring of m single and double edges by n foldings will be denoted by $P(n, m)$.

$P(n,m)$ **graphs**

The $P(n,m)$ graphs with two arc-orbits that are flexibly connected are those with the parameters $m \geq 3$ and $1 \leq n \leq m-1$[3].

They may have have large distinguishing cost.

Theorem *Let $m \geq 5$ and $1 \leq n \leq m-1$.*

Then $\rho(P(n,m)) = \lceil \frac{m}{n} \rceil$,

unless $\lceil \frac{m}{n} \rceil = 2$. Then $\rho = 3$. And, if $m = 3, 4$, then $\rho = 3$.

Let $\rho(G) = \lceil \frac{m}{n} \rceil$. Then $\frac{\rho(G)}{|V(G)|} \approx \frac{m/n}{2^n 2m} = \frac{1}{n2^{n+1}}$

[3] They are also known as Split Praeger–Xu graphs.

The graphs $P(n)$

If we let m in $P(n, m)$ go to ∞ we obtain a graph $P(n)$. It is

- cubic
- vertex transitive
- bipartite
- 2-distinguishable
- has two arc orbits
- is flexibly connected
- positive density is well defined.

The 2-distinguishing density of $P(n)$ is $\frac{1}{n2^{n+1}}$.

Summary of results for v.t. cubic graphs

We have complete answers about cost and density for v.t. cubic graphs with one or three arc orbits.

For v.t. cubic graphs with two arc orbits we have complete answers if they are rigidly connected.

For flexibly connected v.t. cubic graphs we have complete answers for girth three.

For girth four we have many examples of graphs with positive density.

But even in this case we do not know which densities are possible and, e.g. whether there are such graphs with nonlinear growth.

THANK YOU

and

HAPPY BIRTHDAY MARSTON!