Inverses of k-Toeplitz matrices for resonator arrays with multiple receivers

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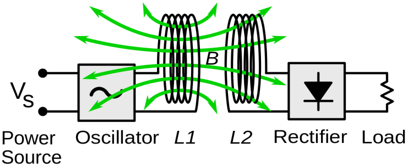
Wireless power transfer

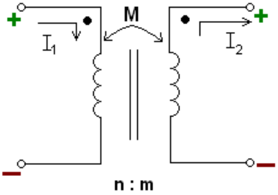
- **Wireless power transfer** (WPT) is used to **transfer energy** while **avoiding electrical contact**
- Useful for **safety** measures and in **rough** environments: water, dust, dirt
- **E.g.:** electrical vehicle charging, biomedical devices

WPT implementation

- **Electromagnetic induction:** power is transferred between coils **through a magnetic field** (by Ampere's law)

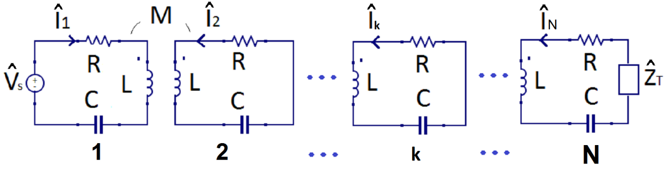


- **Circuit representation:** mutual inductance M



WPT over long distances

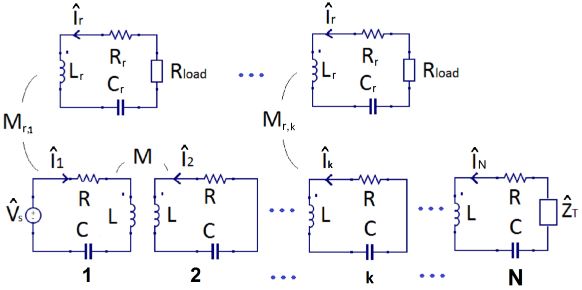
- WPT over longer distances can be achieved via an **array of** parallel **resonators**



- The receiver absorbing the power is placed over some resonator

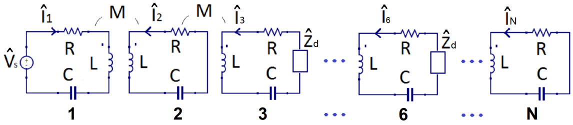
WPT over long distances

- The array could transmit power to **several receivers**
- Cases with more than one receiver were **difficult to analyze** until now
- We study the case with **receivers placed periodically** over the array



Circuit analysis

- We have to analyze an equivalent circuit with **resistors**, **capacitors** and **inductors**, together with their **mutual inductances**, which is periodic



- Given the source voltage and the component parameters, we want to find the **currents** through each component, to compute **power transference** and **efficiency**

Circuit analysis

- The **Fourier transform** allows to analyze the circuit with **linear algebra**
- For a **fixed frequency** ω each component X **behaves like a resistor** in the frecuency domain:

$$V_X = Z_X I_X$$

- Z_X is called the **impedance** of the component X

Component	**Time domain**	**Impedance**
Resistor	$V = RI$	$Z_R = R$
Inductor	$V = L\frac{dI}{dt}$	$Z_L = i\omega L$
Capacitor	$V = \frac{1}{C}\int Idt$	$Z_C = \frac{1}{i\omega C}$

Circuit analysis

- The **current-voltage relations** of the circuit are expressed in the matrix equation

$$AI = V,$$

where A the **impedance matrix**, V the vector of voltage sources, I the vector of unknown currents

- To fully characterize the system **it is enough to invert this matrix**:

$$I = A^{-1}V$$

Matrix of the circuit

- In our case we get a **tridiagonal k-Toeplitz matrix with constant and equal upper and lower diagonals**:

$$A = \begin{pmatrix} Z & i\omega M & 0 & \dots & \dots & \dots & \dots & 0 \\ i\omega M & Z & i\omega M & 0 & \dots & \dots & \dots & 0 \\ 0 & i\omega M & Z+Z_d & i\omega M & 0 & \dots & \dots & 0 \\ 0 & 0 & i\omega M & Z & i\omega M & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & i\omega M \\ 0 & 0 & \dots & \dots & \dots & 0 & i\omega M & Z+Z_T \end{pmatrix}$$

k-Toeplitz matrices

- We define $M_n(a_1, \ldots, a_k, b) \in \mathbb{M}_n(\mathbb{C})$ as

$$\begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{b} & 0 & \ldots & \ldots & 0 \\ \boldsymbol{b} & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \boldsymbol{a_k} & \boldsymbol{b} & \ddots & \vdots \\ \vdots & \ddots & \boldsymbol{b} & \boldsymbol{a_1} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \boldsymbol{b} \\ 0 & \ldots & \ldots & 0 & \boldsymbol{b} & \boldsymbol{a_m} \end{pmatrix}$$

where $m := n \,(\mathrm{mod}\ k)$ and $a_0 := a_k$.

Computing the inverse

- We compute the inverse of $A := M_n(a_1, \ldots, a_k, b)$ **through the adjugate matrix**

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det A}$$

- $\mathrm{adj}(A) = (C_{ij})_{i,j=1}^{n}$ is formed by the **cofactors**

$$C_{ij} := (-1)^{i+j} \det(A_{ij})$$

  (A is symmetric)
- A_{ij} the submatrix formed by removing ith row and jth column

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Application

Computing the inverse: cofactors

- The submatrix A_{ij} is upper block-triangular with three diagonal blocks:

$$A = \left(\begin{array}{c|ccccc|c} A_1 & b & & & & & \\ \hline \boldsymbol{b} & \boldsymbol{a_i} & \boldsymbol{b} & & & & \\ & b & a_{i+1} & b & & & \\ & & b & \ddots & \boldsymbol{b} & & \\ & & & b & \boldsymbol{a_j} & & \\ \hline & & & & \boldsymbol{b} & A_2 \end{array}\right)$$

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- $A_1 = M_{i-1}(a_1, \ldots, a_k, b)$
- $A_2 = M_{n-j}(\sigma_j(a_1, \ldots, a_k), b)$ with σ_j the jth cyclic permutation to the left, $\sigma_0 = \mathrm{id}$
- The middle block B has determinant b^{j-i}

Computing the inverse

- We need $C_{ij} = (-1)^{i+j} \det(A_{ij})$
- The determinant of a block-triangular matrix is **the product of the determinants of its diagonal blocks**

$$\det(A_{ij}) = \det(A_1)\det(B)\det(A_2)$$

- Denote $D_n(a_1, \ldots, a_k, b) := \det(M_n(a_1, \ldots, a_k, b))$
- The (i, j)th element of the inverse A^{-1} is

$$(-b)^{j-1} \frac{D_{i-1}(a_1, \ldots, a_k, b) D_{n-j}(\sigma_j(a_1, \ldots, a_k), b)}{D_n(a_1, \ldots, a_k, b)}$$

Computing the inverse: determinant

- Denote $D(n) := D_n(a_1, \dots, a_k, b)$

$$D(n) = \begin{vmatrix} \boldsymbol{a_1} & \boldsymbol{b} & 0 & \dots & \dots & 0 \\ \boldsymbol{b} & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \boldsymbol{a_k} & \boldsymbol{b} & \ddots & \vdots \\ \vdots & \ddots & \boldsymbol{b} & \boldsymbol{a_1} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \boldsymbol{b} \\ 0 & \dots & \dots & 0 & \boldsymbol{b} & \boldsymbol{a_m} \end{vmatrix}$$

- By **Laplace expansions** along last column and last row

$$D(n) = a_m D(n-1) - b^2 D(n-2)$$

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Computing the inverse: determinant

- We find $D(n)$ as the solution of a **system of k linear recurrence equations**. In matrix form:

$$\begin{pmatrix} D(kn) \\ D(kn-1) \end{pmatrix} = \begin{pmatrix} a_k & -b^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} D(kn-1) \\ D(kn-2) \end{pmatrix},$$

$$\begin{pmatrix} D(kn-1) \\ D(kn-2) \end{pmatrix} = \begin{pmatrix} a_{k-1} & -b^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} D(kn-2) \\ D(kn-3) \end{pmatrix},$$

$$\vdots$$

$$\begin{pmatrix} D(k(n-1)+1) \\ D(k(n-1)) \end{pmatrix} = \begin{pmatrix} a_1 & -b^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} D(k(n-1)) \\ D(k(n-1)-1) \end{pmatrix}.$$

Computing the inverse: determinant

- So
$$\begin{pmatrix} D(kn) \\ D(kn-1) \end{pmatrix} = C^{n-1} \begin{pmatrix} D(k) \\ D(k-1) \end{pmatrix}$$
with
$$C := \begin{pmatrix} a_k & -b^2 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & -b^2 \\ 1 & 0 \end{pmatrix}$$
- The other $k-2$ determinants of the form $D(kn-r)$ are computed from the previous ones
- C^{n-1} is computed by **diagonalization**

Example: For $k = 3$:

- Put $a^3 := a_1 a_2 a_3$, $s := a_1 + a_2 + a_3$, $d(a_1, a_2) := a_1 a_2 - b^2$
- We get

$$\begin{pmatrix} D(3n) \\ D(3n-1) \end{pmatrix} = C^{n-1} \begin{pmatrix} a^3 + (a_2 - s)b^2 \\ d \end{pmatrix}$$

- C has characteristic polynomial $X^2 + (sb^2 - a^3)X + b^6$ with eigenvalues $r_{1,2}$. Computing C^{n-1} we find

$$D(3n) = \frac{1}{r_1 - r_2}((r_1 + a_2 b^2) r_1^n - (r_2 + a_2 b^2) r_2^n),$$
$$D(3n-1) = \frac{d}{r_1 - r_2}(r_1^n - r_2^n).$$

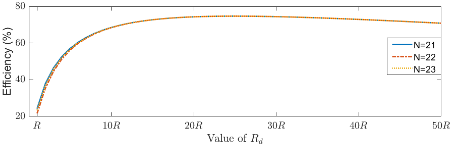
- $D(3n-2) = a_1 D(3(n-1)) - b^2 D(3(n-1)-1)$, so

$$D(3n-2) = \frac{1}{r_1 - r_2}((a_1 r_1 + b^4) r_1^{n-1} - (a_1 r_2 + b^4) r_2^{n-1}).$$

Application

- We get **rational formulas** for the currents, powers, and efficiencies of the WPT system
- **Example:** Efficiency for $N = 3n$, a receptor in each multiple of 3:

$$\eta_{3n} = \frac{R_d(R^2 + (\omega_0 M)^2)}{(r_3^n - r_4^n)(t_3 r_3^{n-1} - t_4 r_4^{n-1})} \sum_{j=1}^{n-1} (\omega_0 M)^{6j-2} (r_3^{n-j} - r_4^{n-j})^2$$



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