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A Fixed Point Approach for Decaying Solutions of Functional Difference Equations

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Joint research with

Zuzana Došlá (Masaryk Univ. Brno) Mauro Marini (University of Florence) **Difference Equations**

functional equations for the unknown $u : \mathbb{N} \to \mathbb{R}$ $f(k, u_k, u_{k+1}, \dots, u_{k+n}) = 0, \ k \in \mathbb{N}$

 Δ : forward difference operator, $\Delta u_k = u_{k+1} - u_k$

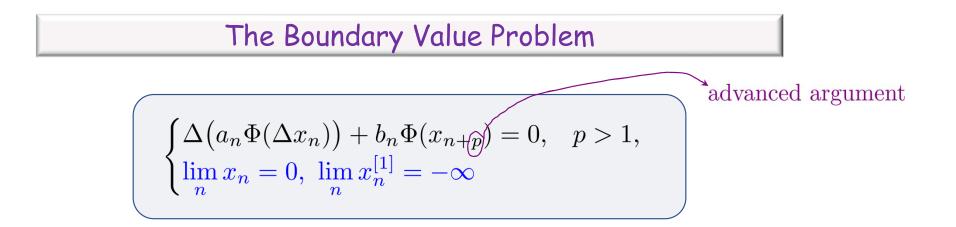
 $F(k, u_k, \Delta u_k, \dots, \Delta^n u_k) = 0$

Appear naturally in mathematical models describing real life situations

(probability theory, queuing problems, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical networks, quanta in radiation, genetics in biology, economics, psychology, sociology, etc)

discrete analogs of differential equation

advent of computers: differential equations are solved by using their approximate difference equation formulations.

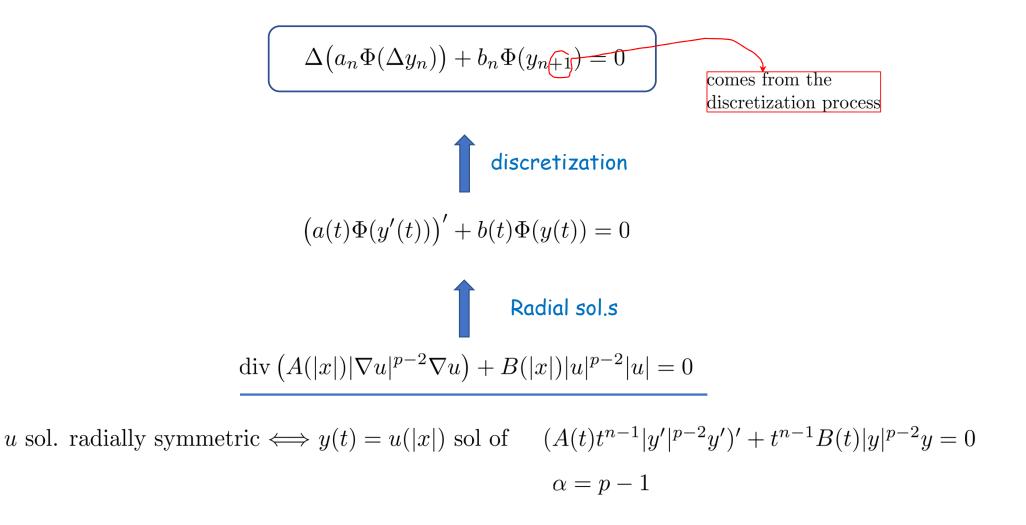


where $\Phi(u) = |u|^{\alpha} \operatorname{sgn} u, \alpha > 0, a, b$ positive sequences, $x_n^{[1]} = a_n \Phi(\Delta x_n) =$ quasi-difference of x

The above equation appears in the discretization process for searching radial solutions of certain nonlinear elliptic equations with weighted φ – laplacian



p = 1: half-linear difference equation



$$\begin{cases} \Delta \left(a_n \Phi(\Delta x_n) \right) + b_n \Phi(x_{n+p}) = 0, \quad p > 1, \\ \lim_n x_n = 0, \ \lim_n x_n^{[1]} = -\infty \end{cases}$$

Assumptions

$$\sum_{i=1}^{\infty} \Phi^* \left(\frac{1}{a_i}\right) < \infty, \quad \sum_{i=1}^{\infty} b_i = \infty$$

a large

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in contrast to the half-linear case, nonoscillatory solutions may coexist with oscillatory (Sturm theory does not hold)

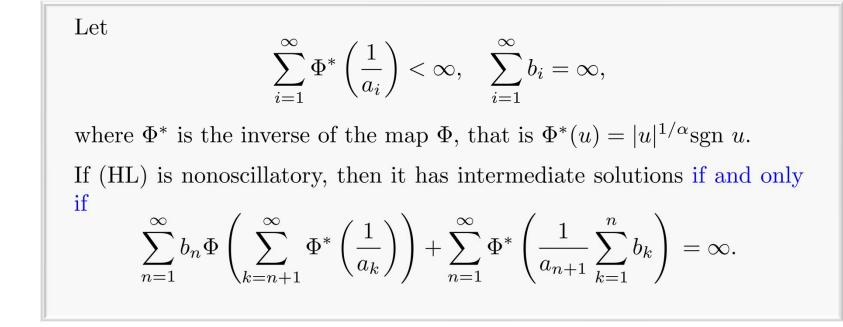
every eventually positive sol. is eventually monotone. Eventually decreasing sol.s satisfy one of the following

$$\lim_{n} x_n = 0, \lim_{n} x_n^{[1]} = -d_x, \ 0 < d_x < \infty \qquad \text{Subdominant sol.}$$

$$\lim_{n} x_n = 0, \lim_{n} x_n^{[1]} = -\infty \qquad \text{Intermediate sol.}$$

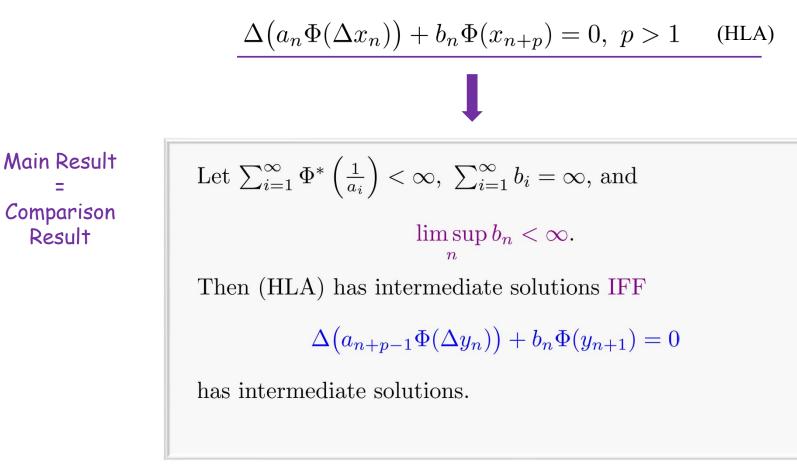
$$\lim_{n} x_n = c, \lim_{n} x_n^{[1]} = -\infty \qquad \text{Dominant sol.}$$

 $p = 1: \text{ half-linear difference equation} \qquad \Delta (a_n \Phi(\Delta y_n)) + b_n \Phi(y_{n+1}) = 0 \qquad (\text{HL})$ \square Problem completely solved



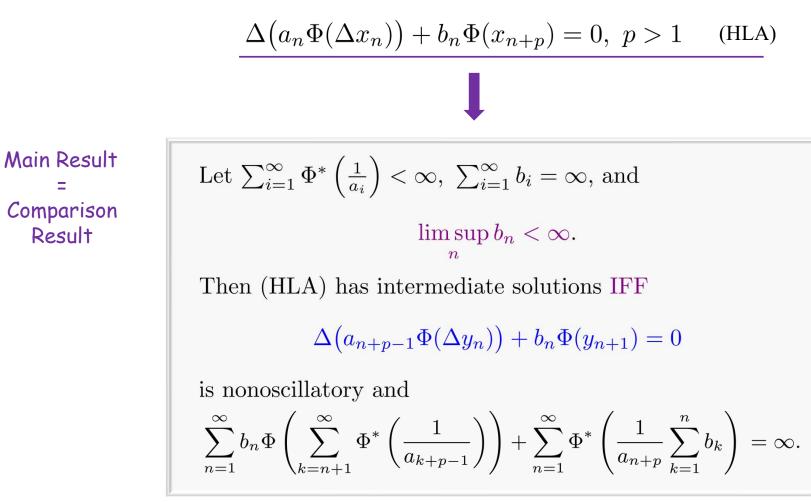


Is it possible to use the characterization of intermediate sol.s for (HL) to obtain existence results for equation with advanced argument?





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Fixed point approach - Topological setting

 $\mathbb{N}_m = \{ i \in \mathbb{N} : i \ge m \in \mathbb{N} \}, \ \mathbb{N}_{m,n} = \{ i \in \mathbb{N}_m : i \le n, m < n \}$

 $\checkmark \quad \mathbb{X} = \{ u : \mathbb{N}_m \to \mathbb{R} \} \quad \text{Frechét space} \text{ with the topology of pointwise convergence on } \mathbb{N}_m$

 $\checkmark \quad \Omega \subset \mathbb{X} \text{ is bounded iff it consists of sequences which are equibounded on } \mathbb{N}_{m,n} \text{ for any } n > m.$ $\longleftrightarrow \quad \exists \Psi_1, \Psi_2 \in \mathbb{X} : \Psi_{1k} \leq u_k \leq \Psi_{2k}, \forall k \geq m, \forall u \in \Omega$

✓ Discrete Arzelà-Ascoli theorem \implies any bounded set in X is relatively compact

 $\land \Omega \subset \mathbb{X} \text{ bounded} \Longrightarrow \Delta \Omega \text{ bounded} \qquad \Delta \Omega = \{\Delta u, u \in \Omega\}$



A Fixed point result

D.M.M. Phil. Trans. R. Soc. A 379 (2021)

Consider the BVP

$$\begin{cases} \Delta(a_n \Phi(\Delta x_n)) = g(n, x), & n \in \mathbb{N}_m \\ x \in S, \end{cases}$$
 (1)

where $g: \mathbb{N}_m \times \mathbb{X} \to \mathbb{R}$ is a continuous map, and S is a subset of \mathbb{X} . Let $G: \mathbb{N}_m \times \mathbb{X}^2 \to \mathbb{R}$ be a continuous map such that $\underline{G(k, u, u)} = \underline{g(k, u)}$ for all $(k, u) \in \mathbb{N}_m \times \mathbb{X}$. If there exist a nonempty, closed, convex set $\Omega \subset \mathbb{X}$, and a bounded, closed subset $S_C \subset S \cap \Omega$ such that the problem

$$\begin{cases} \Delta(a_n \Phi(\Delta x_n)) = G(n, x, q), & n \in \mathbb{N}_m \\ x \in S_C, \end{cases}$$
(2)

has a unique solution for any $q \in \Omega$ fixed, then (1) has at least a solution



Discrete counterpart of Theorem 1.3 in M.Cecchi, M.Furi, M.Marini, *Nonlinear Anal.* 9 (1985)

✓ No explicit form of the fixed point operator is needed

Gives a sufficient condition for the Schauder-Tychoff fixed point theorem to be applicable

Continuity and compactness are consequences of good a-priori bounds

Let $T: \Omega \to S_C$ be the solution operator for (2) S_C bounded $\Rightarrow T(\Omega)$ rel. compact S_C closed $\Rightarrow T$ continuous in Ω

 \checkmark A key point is the choice of the map G

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$$i_{1}) \quad G(n,q,x) = g(n,q), \qquad (i_{2}) \quad G(n,q,x) = \frac{g(n,q)}{\Phi(q_{n+1})} \Phi(x_{n+1})$$

leads to a BVP associated to a second order half-linear difference equation
The theory of half-linear equations can be used to solve
a large variety of BVP. See for instance M. Marini, M.M,
P.Řehák, Adv. Difference Equ. (2006)





A key point is the choice of the map G

 $\begin{array}{ccc} i_1) & G(n,q,x) = g(n,q), \\ G \mbox{ does not depend on } x \implies \mbox{ affine equation. Particularly} \\ \mbox{ useful to solve BVPs associated to difference equations with} \end{array} \qquad i_2) \quad G(n,q,x) = \frac{g(n,q)}{\Phi(q_{n+1})} \Phi(x_{n+1})$

deviating arguments,

Can lead to a BVP associated to a second order difference equation without deviating argument



Let
$$\sum_{i=1}^{\infty} \Phi^*\left(\frac{1}{a_i}\right) < \infty$$
, $\sum_{i=1}^{\infty} b_i = \infty$, and $\limsup_n b_n < \infty$.
Then (HLA) has intermediate solutions IFF (HLp) has intermediate solutions.
(HLA) $\Delta(a_n \Phi(\Delta x_n)) + b_n \Phi(x_{n+p}) = 0$ (HLp) $\Delta(a_{n+p-1} \Phi(\Delta y_n)) + b_n \Phi(y_{n+1}) = 0$

Idea of the proof. $(HLA) \Rightarrow (HLp)$

$$\begin{cases} \Delta (a_{n+p-1}\Phi(\Delta y_n)) + b_n \Phi(y_{n+1}) = 0, \\ \lim_n a_{n_0+p-1}\Phi(\Delta y_n) = -\infty, \\ \lim_n y_n = 0, \end{cases} \longrightarrow \begin{cases} \Delta (a_{n+p-1}\Phi(\Delta z_n)) + b_n \Phi(u_{n+1}) = 0, \\ a_{n_0+p-1}\Phi(\Delta z_{n_0}) = x_{n_0}^{[1]}, \\ \lim_n z_n = 0, \end{cases} \xrightarrow{(n)}$$

 n_0 suff. large: $0 < x_n < 1$, $\Delta x_n < 0$ for $n \ge n_0 \ge 1$ x intermediate sol. of (HLA)



 $\Omega = \left\{ u \in \mathbb{X} : M x_{n+p-1} \le u_n \le x_{n+p-1} \right\}$ nonempty, closed, convex (and bounded)

$$M < 1, \ \Phi(M) \le \frac{|x_{n_0}^{[1]}|}{L + |x_{n_0}^{[1]}|}, \ L \ge \sum_{i=n}^{n+p-2} b_i \ \forall n \ge n_0$$

aux. problem has a unique solution $z = T(u) \subset S = \left\{ v \in \mathbb{X} : a_{n_0+p-1} \Phi(\Delta v_{n_0}) = x_{n_0}^{[1]}, \lim_{n \to \infty} v_n = 0 \right\}$ $T(\Omega) \subset \Omega$

$$S_C = S \cap \Omega = \left\{ v \in \mathbb{X} : a_{n_0+p-1} \Phi(\Delta v_{n_0}) = x_{n_0}^{[1]}, M x_{n+p-1} \le v_n \le x_{n+p-1} \right\} \text{ bounded and closed}$$

aux. problem has a unique solution in $S_C \longrightarrow (\text{HLp})$ has a solution y s.t. $\lim y_n = 0$ $x_{n+p-1}^{[1]} \leq y_n^{[1]} \leq \Phi(M) x_{n+p-1}^{[1]} \longrightarrow \lim_n y_n^{[1]} = -\infty$





$$\Delta (n^{1+\alpha} \Phi(\Delta x_n)) + \gamma \Phi(x_{n+p}) = 0, \quad n \ge p \ge 2$$
 has intermediate solutions IFF
$$0 < \gamma \le \left(\frac{1}{1+\alpha}\right)^{\alpha+1}$$

by means of a change of variable, transform (HLp) into the generalized discrete Euler equation

Comments and Open Problems

- for the case $p \leq 0$ requires a different approach
- **f** condition $\limsup_n b_n < \infty$ is not necessary for existence of intermediate sol.s



does the comparison Thm hold removing this assumption?



Short list of references

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- Zuzana, Došlá; Mauro, Marini; Serena, Matucci (2016). Decaying solutions for discrete coundary value problems on the half line. J. Difference Equ. Appl., vol. 22, pp. 1244-1260.
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