A Fixed Point Approach for Decaying Solutions of Functional Difference Equations

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Difference Equations

functional equations for the unknown $u : \mathbb{N} \to \mathbb{R}$

$$f(k, u_k, u_{k+1}, \ldots, u_{k+n}) = 0, \;\; k \in \mathbb{N}$$

Δ : forward difference operator, $\Delta u_k = u_{k+1} - u_k$

$$F(k, u_k, \Delta u_k, \ldots, \Delta^n u_k) = 0$$

- Appear naturally in mathematical models describing real life situations

  (probability theory, queuing problems, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical networks, quanta in radiation, genetics in biology, economics, psychology, sociology, etc)

- discrete analogs of differential equation

  advent of computers: differential equations are solved by using their approximate difference equation formulations.

The Boundary Value Problem

advanced argument

$$\begin{cases} \Delta\big(a_n\Phi(\Delta x_n)\big) + b_n\Phi(x_{n+p}) = 0, \quad p > 1, \\ \lim_n x_n = 0, \ \lim_n x_n^{[1]} = -\infty \end{cases}$$

where $\Phi(u) = |u|^\alpha \operatorname{sgn} u$, $\alpha > 0$, a, b positive sequences, $x_n^{[1]} = a_n\Phi(\Delta x_n)$ = quasi-difference of x

The above equation appears in the **discretization process for searching radial solutions** of certain nonlinear elliptic equations with weighted $\varphi -$ laplacian

$p = 1$: half-linear difference equation

$$\Delta\big(a_n\Phi(\Delta y_n)\big) + b_n\Phi(y_{n+1}) = 0$$

comes from the discretization process

↑ discretization

$$\big(a(t)\Phi(y'(t))\big)' + b(t)\Phi(y(t)) = 0$$

↑ Radial sol.s

$$\text{div}\,\big(A(|x|)|\nabla u|^{p-2}\nabla u\big) + B(|x|)|u|^{p-2}|u| = 0$$

u sol. radially symmetric $\Longleftrightarrow$ $y(t) = u(|x|)$ sol of $\quad (A(t)t^{n-1}|y'|^{p-2}y')' + t^{n-1}B(t)|y|^{p-2}y = 0$

$$\alpha = p - 1$$

$$\begin{cases} \Delta(a_n\Phi(\Delta x_n)) + b_n\Phi(x_{n+p}) = 0, \quad p > 1, \\ \lim_n x_n = 0, \ \lim_n x_n^{[1]} = -\infty \end{cases}$$

Assumptions

$$\sum_{i=1}^{\infty} \Phi^*\left(\frac{1}{a_i}\right) < \infty, \quad \sum_{i=1}^{\infty} b_i = \infty$$

- in contrast to the half-linear case, nonoscillatory solutions may coexist with oscillatory (Sturm theory does not hold)

- every eventually positive sol. is eventually monotone. Eventually decreasing sol.s satisfy one of the following

$$\lim_n x_n = 0, \ \lim_n x_n^{[1]} = -d_x, \ 0 < d_x < \infty$$ Subdominant sol.

$$\lim_n x_n = 0, \ \lim_n x_n^{[1]} = -\infty$$ Intermediate sol.

$$\lim_n x_n = c, \ \lim_n x_n^{[1]} = -\infty$$ Dominant sol.

no a-priori bounds and their asymptotics can be represented by a large variety of functions.

$p = 1$: half-linear difference equation

$$\Delta\big(a_n \Phi(\Delta y_n)\big) + b_n \Phi(y_{n+1}) = 0 \qquad \text{(HL)}$$

Problem completely solved

Let

$$\sum_{i=1}^{\infty} \Phi^*\left(\frac{1}{a_i}\right) < \infty, \quad \sum_{i=1}^{\infty} b_i = \infty,$$

where Φ^* is the inverse of the map Φ, that is $\Phi^*(u) = |u|^{1/\alpha} \text{sgn } u$.

If (HL) is nonoscillatory, then it has intermediate solutions if and only if

$$\sum_{n=1}^{\infty} b_n \Phi\left(\sum_{k=n+1}^{\infty} \Phi^*\left(\frac{1}{a_k}\right)\right) + \sum_{n=1}^{\infty} \Phi^*\left(\frac{1}{a_{n+1}} \sum_{k=1}^{n} b_k\right) = \infty.$$

Is it possible to use the characterization of intermediate sol.s for (HL) to obtain existence results for equation with advanced argument?

$$\Delta\big(a_n\Phi(\Delta x_n)\big) + b_n\Phi(x_{n+p}) = 0,\ \ p > 1 \qquad \text{(HLA)}$$

Main Result = Comparison Result

Let $\sum_{i=1}^{\infty} \Phi^*\left(\frac{1}{a_i}\right) < \infty$, $\sum_{i=1}^{\infty} b_i = \infty$, and

$$\limsup_n b_n < \infty.$$

Then (HLA) has intermediate solutions IFF

$$\Delta\big(a_{n+p-1}\Phi(\Delta y_n)\big) + b_n\Phi(y_{n+1}) = 0$$

has intermediate solutions.

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$$\Delta\big(a_{n+p-1}\Phi(\Delta y_n)\big) + b_n\Phi(y_{n+1}) = 0$$

is nonoscillatory and

$$\sum_{n=1}^{\infty} b_n\Phi\left(\sum_{k=n+1}^{\infty} \Phi^*\left(\frac{1}{a_{k+p-1}}\right)\right) + \sum_{n=1}^{\infty} \Phi^*\left(\frac{1}{a_{n+p}}\sum_{k=1}^{n} b_k\right) = \infty.$$

Fixed point approach - Topological setting

$$\mathbb{N}_m = \{i \in \mathbb{N} : i \geq m \in \mathbb{N}\}, \; \mathbb{N}_{m,n} = \{i \in \mathbb{N}_m : i \leq n, \, m < n\}$$

- $\mathbb{X} = \{u : \mathbb{N}_m \to \mathbb{R}\}$ Frechét space with the topology of pointwise convergence on $\mathbb{N}_m$
- $\Omega \subset \mathbb{X}$ is bounded iff it consists of sequences which are equibounded on $\mathbb{N}_{m,n}$ for any $n > m$.

  ⟺ $\exists \Psi_1, \Psi_2 \in \mathbb{X} : \Psi_{1k} \leq u_k \leq \Psi_{2k}, \forall k \geq m, \, \forall u \in \Omega$
- Discrete Arzelà-Ascoli theorem $\Longrightarrow$ any bounded set in $\mathbb{X}$ is relatively compact
- $\Omega \subset \mathbb{X}$ bounded $\Longrightarrow$ $\Delta\Omega$ bounded $\qquad \Delta\Omega = \{\Delta u, \, u \in \Omega\}$

A Fixed point result

D.M.M. *Phil. Trans. R. Soc. A* 379 (2021)

Consider the BVP

$$\begin{cases} \Delta(a_n \Phi(\Delta x_n)) = g(n,x), & n \in \mathbb{N}_m \\ x \in S, \end{cases} \tag{1}$$

where $g : \mathbb{N}_m \times \mathbb{X} \to \mathbb{R}$ is a continuous map, and S is a subset of $\mathbb{X}$. Let $G : \mathbb{N}_m \times \mathbb{X}^2 \to \mathbb{R}$ be a continuous map such that $G(k,u,u) = g(k,u)$ for all $(k,u) \in \mathbb{N}_m \times \mathbb{X}$. If there exist a nonempty, closed, convex set $\Omega \subset \mathbb{X}$, and a bounded, closed subset $S_C \subset S \cap \Omega$ such that the problem

$$\begin{cases} \Delta(a_n \Phi(\Delta x_n)) = G(n,x,q), & n \in \mathbb{N}_m \\ x \in S_C, \end{cases} \tag{2}$$

has a unique solution for any $q \in \Omega$ fixed, then (1) has at least a solution

- Discrete counterpart of Theorem 1.3 in M.Cecchi, M.Furi, M.Marini, *Nonlinear Anal*. 9 (1985)

- No explicit form of the fixed point operator is needed

- Gives a sufficient condition for the Schauder-Tychoff fixed point theorem to be applicable

- Continuity and compactness are consequences of good a-priori bounds

Let $T : \Omega \to S_C$ be the solution operator for (2)

$$S_C \text{ bounded} \Rightarrow T(\Omega) \text{ rel. compact}$$
$$S_C \text{ closed} \Rightarrow T \text{ continuous in } \Omega$$

- A key point is the choice of the map G

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$$i_1)\quad G(n,q,x) = g(n,q), \qquad i_2)\quad G(n,q,x) = \frac{g(n,q)}{\Phi(q_{n+1})}\Phi(x_{n+1})$$

leads to a BVP associated to a second order half-linear difference equation

The theory of half-linear equations can be used to solve a large variety of BVP. See for instance M. Marini, M.M, P.Řehák, *Adv. Difference Equ.* (2006)

A key point is the choice of the map G

$$i_1)\quad G(n,q,x) = g(n,q), \qquad i_2)\quad G(n,q,x) = \frac{g(n,q)}{\Phi(q_{n+1})}\Phi(x_{n+1})$$

G does not depend on $x \implies$ affine equation. Particularly useful to solve BVPs associated to difference equations with deviating arguments.

Can lead to a BVP associated to a second order difference equation without deviating argument

Choice for this problem

Let $\sum_{i=1}^{\infty} \Phi^*\left(\frac{1}{a_i}\right) < \infty$, $\sum_{i=1}^{\infty} b_i = \infty$, and $\limsup_n b_n < \infty$.

Then (HLA) has intermediate solutions IFF (HLp) has intermediate solutions.

(HLA) $\Delta\big(a_n\Phi(\Delta x_n)\big) + b_n\Phi(x_{n+p}) = 0$ (HLp) $\Delta\big(a_{n+p-1}\Phi(\Delta y_n)\big) + b_n\Phi(y_{n+1}) = 0$

Idea of the proof. (HLA) $\Rightarrow$ (HLp)

$$\begin{cases}\Delta\big(a_{n+p-1}\Phi(\Delta y_n)\big) + b_n\Phi(y_{n+1}) = 0, \\ \lim_n a_{n_0+p-1}\Phi(\Delta y_n) = -\infty, \ \lim_n y_n = 0,\end{cases} \Longrightarrow \begin{cases}\Delta\big(a_{n+p-1}\Phi(\Delta z_n)\big) + b_n\Phi(u_{n+1}) = 0, \quad n \geq n_0 \\ a_{n_0+p-1}\Phi(\Delta z_{n_0}) = x_{n_0}^{[1]}, \ \lim_n z_n = 0,\end{cases}$$

n_0 suff. large: $0 < x_n < 1$, $\Delta x_n < 0$ for $n \geq n_0 \geq 1$

x intermediate sol. of (HLA)

$\Omega = \{u \in \mathbb{X} : M x_{n+p-1} \le u_n \le x_{n+p-1}\}$ nonempty, closed, convex (and bounded)

$$M < 1,\ \Phi(M) \le \frac{|x_{n_0}^{[1]}|}{L + |x_{n_0}^{[1]}|},\ L \ge \sum_{i=n}^{n+p-2} b_i \ \forall n \ge n_0$$

- aux. problem has a unique solution $z = T(u) \subset S = \left\{v \in \mathbb{X} :\ a_{n_0+p-1}\Phi(\Delta v_{n_0}) = x_{n_0}^{[1]},\ \lim_n v_n = 0\right\}$
- $T(\Omega) \subset \Omega$

$S_C = S \cap \Omega = \left\{v \in \mathbb{X} :\ a_{n_0+p-1}\Phi(\Delta v_{n_0}) = x_{n_0}^{[1]},\ M x_{n+p-1} \le v_n \le x_{n+p-1}\right\}$ bounded and closed

aux. problem has a unique solution in S_C ⟶ (HLp) has a solution y s.t. $\lim y_n = 0$

$x_{n+p-1}^{[1]} \le y_n^{[1]} \le \Phi(M) x_{n+p-1}^{[1]}$ ⟶ $\lim_n y_n^{[1]} = -\infty$

An Example

$$\Delta\big(n^{1+\alpha}\Phi(\Delta x_n)\big) + \gamma\,\Phi(x_{n+p}) = 0, \quad n \ge p \ge 2$$

has intermediate solutions IFF

$$0 < \gamma \le \left(\frac{1}{1+\alpha}\right)^{\alpha+1}$$

by means of a change of variable, transform (HLp) into the generalized discrete Euler equation

Comments and Open Problems

- the case $p \le 0$ requires a different approach
- condition $\limsup_n b_n < \infty$ is not necessary for existence of intermediate sol.s

→ does the comparison Thm hold removing this assumption?

Short list of references

- Zuzana Došlá; Mauro Marini; Serena Matucci (2021). A fixed-point approach for decaying solutions of difference equations. *Phil. Trans. R. Soc. A*. http://doi.org/10.1098/rsta.2019.0374
- Zuzana Došlá; Serena Matucci; Pavel Řehák (2020). Decaying positive global solutions of second order difference equations with mean curvature operator. *Electron. J. Qual. Theory Differ. Equ.*, No. 72, 1-16. https://doi.org/10.14232/ejqtde.2020.1.72
- Zuzana, Došlá; Mauro, Marini; Serena, Matucci (2016). Decaying solutions for discrete boundary value problems on the half line. *J. Difference Equ. Appl*., vol. 22, pp. 1244-1260. https://doi.org/10.1080/10236198.2016.1190349

Thanks for your attention!