PNDP-manifolds

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Abstract

We present a new type of manifold called partially negative dimensional product manifold (PNDP-manifold for short). In particular a PNDP-manifold is a special type of Einstein sequential warped product manifold, where the base-manifold B is a Remannian (or pseudo-Riemannian) product-manifold $B = B_1 \times B_2$ (where B_1 is an Einstein-manifold), and the fiber-manifold F is a derived smooth manifold (i.e., F is the Kuranishi neighborhood ($\mathbb{R}^d$, E, S), where E is the bundle of obstruction and S is a smooth section, so it can admit a "virtual" dimension which can also be negative).
From differential geometric point of view, this special type of Einstein sequential warped product manifold allows to cover a wider variety of exact solutions of Einstein's field equation, without complicating the calculations much, compared to the Einstein warped-product manifolds with Ricci-flat fiber $(F; \ddot{g})$. From speculative point of view, considering the fiber as derived smooth manifold, the dimensions of a PNDP-manifold is not related with the usual geometric concept of dimension (we consider them as "virtual" dimensions), and with a correct interpretation it is possible to consider a new type of "hidden" dimensions that lead to many speculative/applicative aspects, such as in the econophysical field, in the description of financial markets influenced by ghost fields as dark volatility, but also in cosmological field introducing the concept of "emerging spaces".

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1. Introduction

The concept of negative dimensional space is not new, in fact it is already used in linguistic statistics [1] and also in supersymmetric theories in Quantum Field Theory, [2].
Here is considered a special type of Einstein sequential warped product manifold, where as a "mathematical tool", for the fiber-manifold (F), it uses the derived geometry and about this, let's make a small consideration. If $A \to M$ and $B \to M$ are two transversal submanifolds of codimension a and b respectively, then their intersection C is again a submanifold, of codimension $a+b$. Derived geometry, for example, explains how to remove the transversality condition and make sense out of a nontransversal intersection C as a derived smooth manifold of codimension $a+b$. In particular $dim(C) = dim(M) - a - b$ and therefore the latter number can also be negative. In this way the obtained dimension are not related to the usual geometrical concept of "dimension", but it can be considered a "virtual" dimension.
In this paper we will consider F as a derived smooth manifold of the Kuranishi neighborhood type ($\mathbb{R}^d$, E, S).

2. Preliminaries

Let's start by providing a brief general introduction to warped product manifolds, a type of manifold introduced by Bishop and O'Neill in [3], to construct a wide variety of manifolds of negative curvature. For further insights into this classic topic we recommend [3] and [4].

Let (B, g_B) and (F, g_F) be two Riemannian manifolds and τ, σ be the projection of $B \times F$ onto B and F, respectively.

Definition 2.1. *The warped product $M = B \times_f F$ is the manifold $B \times F$ equipped with the metric tensor $g = \tau^* g_B + f^2 \sigma^* g_F$, where * denotes the pullback and f is a positive smooth function on B, the so-called warping function.*

Explicitly, if X is tangent to $B \times F$ at (p, q) (where p is a point on B and q is a point on F), then:

$$\langle X, X\rangle = \langle d\tau(X), d\tau(X)\rangle + f^2(p)(d\sigma(X), d\sigma(X)). \tag{2.1}$$

B is called the base-manifold of $M = B \times_f F$ and F is the fiber-manifold; If $f = 1$, then $B \times_f F$ reduces to a Riemannian product manifold. The leaves $B \times q = \Sigma^{-1}(q)$

and the fibers $p \times F = \tau^{-1}(p)$ are Riemannian submanifolds of M. Vectors tangent to leaves are called horizontal and those tangent to fibers are called vertical. $\mathcal{H}$ will be the orthogonal projection of $T_{(p,q)}M$ onto its horizontal subspace $T_{(p,q)}(B \times q)$ and $\mathcal{V}$ will be the projection onto the vertical subspace $T_{(p,q)}(p \times F)$.

Lemma 2.1. *If $h \in \mathcal{F}(B)$, where $\mathcal{F}(B)$ is the set of all smooth real-valued functions on M, then the gradient of the lift $h \circ \tau$ of h to $M = B \times_f F$ is the lift to M of the gradient of h on B.*

Proof. If v is a vertical tangent vector to M, then $\langle \nabla(h \circ \tau), v \rangle = v(h \circ \tau) = d\tau(v)h = 0$, since $d\tau(v) = 0$. Thus $\nabla(h \circ \tau)$ is horizontal. If w is horizontal,

$$\langle d\tau(\nabla(h \circ \tau)), d\tau(w) \rangle = \langle \nabla(h \circ \tau), w \rangle = w(h \circ \tau) = d\tau(w)h = \langle \nabla h, d\tau(w) \rangle, \tag{2.2}$$

which implies that at each point, $d\tau(\nabla(h \circ \tau)) = \nabla h$. □

The following proposition is to define the Levi-Civita connection ∇ of $M = B \times_f F$ related to those of B and F.

Proposition 2.1. *On $M = B \times_f F$, if $X, Y \in \mathcal{L}(B)$ and $U, V \in \mathcal{L}(F)$, (where $\mathcal{L}(B)$ is the set of all lifts on B to $B \times F$ and $\mathcal{L}(F)$ is the set of all lifts on F to $B \times F$), then:*
(a) $\nabla_X Y \in \mathcal{L}(B)$ is the lift of $\nabla_X Y$ on B,
(b) $\nabla_X U = \nabla_U X = (X ln f)U$,
(c) $nor(\nabla_U V) = \eta(U, V) = -\frac{\langle U,V \rangle}{f} \nabla f$, where $\eta(U, V)$ is the second fundamental form on the fiber,
(d) $tan(\nabla_U V) \in \mathcal{L}(F)$ is the lift of $\ddot{\nabla}_U V$ on F, where $\ddot{\nabla}$ is the Levi-Civita connection of F.

Proof. The Koszul formula for $2\langle \nabla_X Y, U \rangle$ reduces to $\langle U, [X, Y] \rangle - U\langle X, Y \rangle$ due to $[X, U] = [Y, U] = 0$. Since X, Y are lift from B, then $\langle X, Y \rangle$ is constant on fibers. Because U is vertical, $U\langle X, Y \rangle = 0$. But $[X, Y]$ is tangent to leaves, so $\langle U, [X, Y] \rangle = 0$. Thus $\langle \nabla_X Y, U \rangle = 0$ for all $U \in \mathcal{L}(F)$. This shows that $\nabla_X Y$ is horizontal and since $\tau|_{B \times q}$ is an isometry, (a) is obtained.
From $[X, U] = 0$, we find $\nabla_X U = \nabla_U X$. Since these vector fields are vertical, then $\langle \nabla_X U, Y \rangle = -\langle U, \nabla_X Y \rangle = 0$. All the terms in the Koszul formula for $2\langle \nabla_X U, V \rangle$ vanish except $X\langle U, V \rangle$.

So:

$$(e) 2\langle \nabla_X U, V\rangle = X\langle U, V\rangle. \tag{2.3}$$

By the definition of warped product metric, we find $\langle U, V\rangle_{(p,q)} = f^2(p)\langle U_q, V_q\rangle$. After writing f for $f\circ\tau$, we have $\langle U, V\rangle = f^2((U,V)\circ\sigma)$. Hence $X\langle U, V\rangle = X[f^2(U,V)\circ\sigma)] = 2fX[f((U,V)\circ\sigma)] = 2(Xlnf)\langle U, V\rangle$. Combing this with (e) gives property (b), and the property (b) implies:

$$\langle \nabla_U V, X\rangle = -\langle V, \nabla_U X\rangle = -(Xlnf)\langle U, V\rangle. \tag{2.4}$$

Thus, after applying Lemma 2.1, we find $Xf = \langle \nabla f, X\rangle$ on M as on B. Hence, for any X, we obtain:

$$\langle \nabla_U V, X\rangle f = -\langle U, V\rangle\langle \nabla f, X\rangle, \tag{2.5}$$

which implies property (c).
Since U and V are tangent to all fibers, $tan(\nabla_U V)$ is the fiber covariant derivative applied to the restrictions of U and V to that fiber. Therefore, we have property (d). □

The next result provides the curvature ($\bar{Riem}$) of a warped product $M = B \times_f F$ in terms of its warping function f and the curvature tensors $Riem$ and $\ddot{Riem}$ of B and F respectively.

Proposition 2.2. *Let $M = B \times_f F$ be a warped product of two (pseudo-)Riemannian manifolds. If $X, Y, Z \in \mathcal{L}(B)$ and $U, V, W \in \mathcal{L}(F)$, then:*
(f) $\bar{Riem}(X,Y)Z \in \mathcal{L}(B)$ is the lift of $Riem(X,Y)Z$ on B
(g) $\bar{Riem}(X,V)Y = \frac{H^f(X,Y)}{f}V$,
(h) $\bar{Riem}(X,Y)V = \bar{Riem}(V,W)X = 0$,
(i) $\bar{Riem}(X,V)W = -\frac{\langle V,W\rangle}{f}\nabla_X(\nabla f)$,
(j) $\bar{Riem}(V,W)U = \ddot{Riem}(V,W)U + \frac{\langle \nabla f, \nabla f\rangle}{f^2}\{\langle V,U\rangle W - \langle W,U\rangle V\}$,
where H^f is the Hessian of f.

Proof. The projection $\tau : M \to B$ is isometric on each leaf, so $Riem$ gives the Riemannian curvature tensor of each leaf. Because leaves are totally geodesic in M, $Riem$ agrees with the curvature tensor $\bar{Riem}$ of M on horizontal vectors. Thus we have (f).
Considering $X, Y \in \mathcal{L}(B)$ and $V \in \mathcal{L}(F)$, we have $[X, Y] = 0$ and $Riem(X,V)Y = \nabla_X\nabla_V Y - \nabla_V\nabla_X Y$.

By Proposition 2.1

$$\nabla_X\nabla_V Y = \nabla_X(Y(lnf)V) = (XY(lnf))V + (Y(lnf))\nabla_X V = \tag{2.6}$$

$$= \{XY(lnf) + (Yf)Xf^{-1}\}V + (X(lnf))(Y(lnf))V = \tag{2.7}$$

$$= (XY(lnf))V. \tag{2.8}$$

Thus:

$$(k)\bar{Riem}(X,V)Y = (XY(lnf))V - \nabla_V\nabla_X Y. \tag{2.9}$$

But on the other hand, since $\nabla_X Y \in \mathcal{L}(B)$, we obtain $\nabla_V\nabla_X Y = (\nabla_X Y(lnf))V$, and combining this with (k) gives (g). Now to prove (h) assume that $[V,W]=0$,

$$\nabla_V\nabla_W X = (VX(lnf))W + (X(lnf))\nabla_V W, \tag{2.10}$$

but $X(lnf)$ is coinstant on fibers, so $VX(lnf)=0$. Thus

$$\bar{Riem}(V,W)X = (X(lnf))(\nabla_W V - \nabla_V W) = X(lnf)[W,V] = 0. \tag{2.11}$$

By the symmetry of curvature, $\langle \bar{Riem}(X,Y)V,W\rangle = \langle \bar{Riem}(V,W)X,Y\rangle = 0$. From (f) $\langle \bar{Riem}(X,Y)V,Z\rangle = -\langle \bar{Riem}(X,Y)Z,V\rangle = 0$. These equations hold for all $W \in \mathcal{L}(F)$ and $Z \in \mathcal{L}(B)$, hence $\bar{Riem}(X,Y)V = 0$.
To prove (i) first we note that $\bar{Riem}(X,V)W$ is horizontal, since

$$\langle \bar{Riem}(X,V)W,U\rangle = \langle \bar{Riem}(W,U)X,V\rangle = 0 \tag{2.12}$$

according to (h). Since $\bar{Riem}(V,W)X=0$, it follows from first Bianchi identity

$$\bar{Riem}(X,V)W = \bar{Riem}(X,W)V. \tag{2.13}$$

But using (g):

$$\langle \bar{Riem}(X,V)W,Y\rangle = \langle \bar{Riem}(V,X)Y,W\rangle = -\frac{H^f(X,Y)}{f}\langle V,W\rangle = \tag{2.14}$$

$$-\frac{\langle V,W\rangle}{f}\langle \nabla_X(\nabla f),Y\rangle. \tag{2.15}$$

Since $\bar{Riem}(X,V)W$ is horizontal and the equation holds for all Y, we obtain (i). For (j), we observe that $\bar{Riem}(V,W)U$ is a vertical vector field since $\langle \bar{Riem}(V,W)U,X\rangle = -\langle \bar{Riem}(V,W)X,U\rangle = 0$ by (h). Because the projection $\sigma : M \to F$ is a homothety on fibers, $\ddot{Riem}(v,W)U \in \mathcal{L}(F)$ is the application to V,W,U of the curvature tensor of each fiber. Consequentely, $\ddot{Riem}(V,W)U$ and $\bar{Riem}(V,W)U$ are related by the equation of Gauss. Combining this with the fact that the shape tensor of the fibers is given by $\eta(V,W) = -\frac{\langle V,W\rangle}{f}\nabla f$, we obtain (j). $\square$

Definition 2.2. *Let Riem be the Riemannian curvature tensor of M. The Ricci curvature tensor Ric of M is the contraction $C_3^1(Riem) \in \mathcal{I}_2^0(M)$ (where $\mathcal{I}_s^r(M)$ is the set of all tensor fields on M), whose components relative to a coordinate system are $R_{ij} = \Sigma R_{ijm}^m$. Because of the symmetries of Riem the only nonzero contractions of Riem are $\pm Ric$.*

Lemma 2.2. *The Ricci curvature tensor Ric is symmetric, and is given relative to a frame field by:*

$$Ric(X,Y) = \Sigma_m \varepsilon_m \langle Riem(Y,X)E_m, E_m\rangle, \tag{2.16}$$

where as usual $\varepsilon_m = \langle E_m, E_m\rangle$.

Proof. $Ric(X,Y) = (C_3^1 Riem)(X,Y) = \Sigma \varepsilon_m \langle E_m, Riem(X,Y)E_m\rangle$. Symmetry by pairs then gives the required formula and shows that Ric is symmetric. □

If its Ricci tensor is identically zero, M is said to be Ricci flat; A flat manifold is certainly Ricci flat.

Proposition 2.3. *On a warped product $M = B \times_f F$ with $d = dim(F) > 1$, let X, Y be horizontal vectors and V, W vertical vectors. Then the Ricci tensor $\bar{Ric}$ of M satisfies:*
(k) $\bar{Ric}(X,Y) = Ric(X,Y) - \frac{d}{f}H^f(X,Y)$,
(l) $\bar{Ric}(X,V) = 0$,
(m) $\bar{Ric}(V,W) = \ddot{Ric}(V,W) - \langle V,W\rangle f^*$,
where $f^* = \frac{\Delta f}{f} + (d-1)\frac{\langle \nabla f, \nabla f\rangle}{f^2}$, *Ric and $\ddot{Ric}$ are the lifts of the Ricci curvature of B and of F, respectively.*

The proof is an exercise in tensor computation, just apply the Ricci formula in Lemma 2.2 to a frame field on M whose vector fields are in $\mathcal{L}(B)$ and $\mathcal{L}(F)$.
Now it is possible to provide the classic definition of Einstein-warped product manifold (for more details see eg [5]):

Definition 2.3. *A warped product manifold $(M, \bar{g}) = (B, g) \times_f (F, \ddot{g})$, with metric tensor $\bar{g} = g + f^2\ddot{g}$, is Einstein if only if:*

$$\bar{Ric} = \lambda \bar{g} \iff \begin{cases} Ric - \frac{d}{f}\nabla^2 f = \lambda g \\ \ddot{Ric} = \mu \ddot{g} \\ f\Delta f + (d-1)|\nabla f|^2 + \lambda f^2 = \mu \end{cases} \tag{2.17}$$

where λ and μ are constants, d is the dimension of F, $\nabla^2 f$, Δf and ∇f are, respectively, the Hessian, the Laplacian and the gradient of f for g, with $f : (B) \to \mathbb{R}^+$ a smooth positive function.

Contracting first equation of (2.17) we get:

$$Rf^2 - f\Delta f d = nf^2\lambda \tag{2.18}$$

where n and R is the dimension and the scalar curvature of B respectively, and from third equation, considering $d \neq 0$ and $d \neq 1$, we have:

$$f\Delta f d + d(d-1)|\nabla f|^2 + \lambda f^2 d = \mu d. \tag{2.19}$$

3. Definition and construction of a PNDP-manifold

In the previous section a general introduction to the theory of warped product manifolds was given. In this session we will deal with PNDP-manifolds, special types of Einstein sequential warped product manifolds, introduced by A. Pigazzini et al. in [6], whose base-manifold (B) is a Riemannian product with precise characteristics and whose fiber-manifold (F) is a derived smooth manifold that allows a concept of "virtual" dimensions. For more information about Kuranishi neighborhoods (derived smooth manifolds and obstruction bundle) see [7].

Definition 3.1. *We called PNDP-manifold a warped product manifold* $(M, \bar{g}) = (B, g) \times_f (F, \ddot{g})$ *that satisfies (2.17), where the base-manifold* (B, g) *is a Riemannian (or pseudo-Riemannian) product-manifold* $B = B_1 \times B_2$ *with* $g = \Sigma g_i$*, where* B_2 *is an Einstein manifold (i.e.,* $Ric_2 = \lambda g_2$ *where* λ *is the same for (2.17) and* g_2 *is the metric for* B_2*), with* $dim(B_1) = n_1$*,* $dim(B_2) = n_2$*, so* $dim(B) = n = n_1 + n_2$*. The warping function* $f : B \to \mathbb{R}^+$ *is* $f(x, y) = f_1(x) + f_2(y)$ *(where each is a function on its individual manifold, i.e.,* $f_1 : B1 \to \mathbb{R}^+$ *and* $f_2 : B_2 \to \mathbb{R}^+$*) and can also be a constant function. The fiber-manifold* $(F, \ddot{g})$ *is a derived Riemann-flat manifold with negative "virtual" integer dimensions* m*, where with derived smooth manifold is considered a smooth Riemannian flat manifolds by adding a vector bundle of obstructions. In particular we consider, for* F*, only as* $\mathbb{R}^d$*, with orthogonal Cartesian coordinates such that* $g_{ij} = -\delta_{ij}$*, by adding a vector bundle of obstructions,* $E \to \mathbb{R}^d$*, with dimension* $m = d - rank(E)$*, where* $rank(E) = 2d$*. In fact, in this circumstance, if we consider a Kuranishi neighborhood* $(\mathbb{R}^d, E, S)$*, with manifold* $\mathbb{R}^d$*, obstruction bundle* $E \to \mathbb{R}^d$*, and section* $S : \mathbb{R}^d \to E$*, then the dimension of the derived smooth manifold* F *is* $dim(\mathbb{R}^d) - rank(E)$*. Moreover in the case* $n - d > 0$ *(i.e., positive "virtual" dimension) we consider* $n_1 = d = -m$ *(the "virtual" dimension of* M*,* $dim(M)_{\mathcal{V}}$*, which coincides with* $dim(B_2)$*). In the specal case where* $n - d > 0$ *with also* B_1 *an Einstein-manifold with the same Einstein-*λ*, then we consider only the case* $B_1 = B_2$*.*

Important Note: Since $F := (\mathbb{R}^d, E, S)$, and on E (obstruction bundle) the (pseudo-)Riemannian geometry does not work, each (pseudo-)Riemannian geometry operation is performed and defined only on the underlying $\mathbb{R}^d$, but is considered performed and defined also on F (i.e., for example, we will say that the Ricci curvature of F is zero because the Ricci curvature of $\mathbb{R}^d$ is zero). Being, therefore, that the usual (pseudo-)Riemannian geometry works for the underlying smooth manifold (because it is an ordinary manifolds), from now on it we will work with the (pseudo-)Riemannian geometry on the derived fibers-manifold F, going to define all (pseudo-)Riemannian geometry operations not directly on F, but on $\mathbb{R}^d$, but considering them made on F, paying attention only to the dimension. Obviously for what has been said, the tangent space and the vector fields are those of $\mathbb{R}^d$. The scalar product with two arbitrary vector fields $\ddot{g}\langle V, W\rangle$ is define on F as: $g_{ij}v^i w^j = -\delta_{ij}v^i w^j = -(v^i w^i)$.

The analysis does not differ from the usual Einstein sequential warped product manifold analysis, $(M_1 \times_h M_2) \times_{\bar{h}} M_3$, (see [8], [9]), where $h = 1$, M_2 is an Einstein-manifold and M_3 is a derived-smooth-manifold with negative "virtual" dimensions. The Riemannian curvature tensor and the Ricci curvature tensor of the product Riemannian manifold can be written respectively as the sum of the Riemannian curvature tensor and the Ricci curvature tensor of each Riemannian manifold (see [10]).

Proposition 3.1. *If we write the B-product as $B = B_1 \times B_2$, where:*
i) Ric_i is the Ricci tensor of B_i referred to g_i, where $i = 1, 2$,
ii) $f(x, y) = f_1(x) + f_2(y)$, is the smooth warping function, where $f_i : B_i \to \mathbb{R}^+$,
iii) $Hess(f) = \Sigma_i \tau_i^* Hess_i(f_i)$ is the Hessian referred on its individual metric, where τ_i^* are the respective pullbacks, (and $\tau_2^* Hess_2(f_2) = 0$ since B_2 is Einstein),
iv) ∇f is the gradient (then $|\nabla f|^2 = \Sigma_i |\nabla_i f_i|^2$), and
v) $\Delta f = \Sigma_i \Delta_i f_i$ is the Laplacian, (from (iii) therefore also $\Delta_2 f_2 = 0$).
Then the Ricci curvature tensor will be:

$$\begin{cases} \bar{Ric}(X_i, X_j) = Ric_1(X_i, X_j) - \frac{d}{f} Hess_1(f_1)(X_i, X_j) \\ \bar{Ric}(Y_i, Y_j) = Ric_2(Y_i, Y_j) \\ \bar{Ric}(U_i, U_j) = \ddot{Ric}(U_i, U_j) - \ddot{g}(U_i, U_j) f^* \\ \bar{Ric}(X_i, Y_j) = 0 \\ \bar{Ric}(X_i, U_j) = 0, \\ \bar{Ric}(Y_i, U_j) = 0, \end{cases} \tag{3.1}$$

where $f^* = \frac{\Delta_1 f_1}{f} + (d-1)\frac{|\nabla f|^2}{f^2}$, *and* X_i, X_j, Y_i, Y_j, U_i, U_j *are vector fields on* B_1, B_2 *and* F, *respectively.*

Also in this case the proof is an exercise, just recompute Proposition 2.1 considering the characteristics in Definition 3.1 of the base-manifold $B = B_1 \times B_2$ *and of the warping function* $f(x,y) = (f_1(x) + f_2(y))$. *The Riemann curvature tensors will therefore be redefined by considering the recomputation of Proposition 2.1 and then for Lemma 2.2 we will obtain the system (3.1).*

Theorem 3.1. *A warped product manifold with derived differential fiber-manifold* $F := (\mathbb{R}^d, E, S)$, *and dim*$(F)$ *a negative integer, is a PNDP-manifold, as defined in Definition 3.1, if and only if:*

$$\bar{Ric} = \lambda \bar{g} \Longleftrightarrow \begin{cases} Ric_1 - \frac{d}{f}\tau_1^* \nabla_1^2 f_1 = \lambda g_1 \\ \tau_2^* \nabla_2^2 f_2 = 0 \\ Ric_2 = \lambda g_2 \\ \ddot{Ric} = 0 \\ f\Delta_1 f_1 + (d-1)|\nabla f|^2 + \lambda f^2 = 0, \end{cases} \tag{3.2}$$

(since Ric is the Ricci curvature of B, *then* $Ric = Ric_1 + Ric_2 = \lambda(g_1 + g_2) + \frac{d}{f}\tau_1^* \nabla_1^2 f_1$). *Therefore equations (2.18) and (2.19), for* $n - d = 0$ *and* $n - d < 0$, *become:*

$$\bar{R} = \lambda \bar{n} \Longleftrightarrow \begin{cases} R_1 f - \Delta_1 f_1 d = n_1 f \lambda \\ \Delta_2 f_2 = 0 \\ R_2 = \lambda n_2 \\ \ddot{Ric} = 0 \\ f\Delta_1 f_1 + (d-1)|\nabla f|^2 + \lambda f^2 = 0. \end{cases} \tag{3.3}$$

where n_1 *and* R_1 *are the dimension and the scalar curvature of* B_2 *respectively, while for* $n - d > 0$, *we must set* $d = n_1$. *We have*

$$\bar{R} = \lambda \bar{n} \Longleftrightarrow \begin{cases} R_1 f - \Delta_1 f_1 n_1 = n_1 f \lambda \\ \Delta_2 f_2 = 0 \\ R_2 = \lambda n_2 \\ \ddot{Ric} = 0 \\ f\Delta_1 f_1 + (n_1 - 1)|\nabla f|^2 + \lambda f^2 = 0. \end{cases} \tag{3.4}$$

Proof. We applied the condition that the warped product manifold of the system (3.1) in Proposition 3.1 is Einstein. From this we obtain that for a PNDP-manifold the system (2.17) becomes the system (3.2). □

We underline that from first equation of the systems, R_1 cannot depend on points on B_2, so this imposes a condition on f.

Remark 3.1. *In the particular case where* $d = 1$ *the systems (3.3) and (3.4) are to be modified, in fact for* $d = 1$ *from system (3.2) we get:*

$$\bar{Ric} = \lambda \bar{g} \Longleftrightarrow \begin{cases} Ric_1 - \frac{1}{f}\tau_1^* \nabla_1^2 f_1 = \lambda g_1 \\ \tau_2^* \nabla_2^2 f_2 = 0 \\ Ric_2 = \lambda g_2 \\ \ddot{Ric} = 0 \\ f\Delta_1 f_1 + \lambda f^2 = 0, \end{cases} \tag{3.5}$$

from which system (3.3) becomes:

$$\bar{R} = \lambda \bar{n} \Longleftrightarrow \begin{cases} R_1 f - \Delta_1 f_1 = n_1 f \lambda \\ \Delta_2 f_2 = 0 \\ R_2 = \lambda n_2 \\ \ddot{Ric} = 0 \\ f\Delta_1 f_1 + \lambda f^2 = 0. \end{cases} \tag{3.6}$$

and system (3.4) becomes:

$$\bar{R} = \lambda \bar{n} \Longleftrightarrow \begin{cases} R_1 f - \Delta_1 f_1 = f \lambda \\ \Delta_2 f_2 = 0 \\ R_2 = \lambda n_2 \\ \ddot{Ric} = 0 \\ f\Delta_1 f_1 + \lambda f^2 = 0. \end{cases} \tag{3.7}$$

Recapitulating, the Derived-geometry is used to define the fiber-manifold and therefore admit the presence of negative "virtual" dimensions. Since, from the point of view of (pseudo-)Riemannian geometric operations, we consider the fiber-manifold F as $\mathbb{R}^d$, then the classical construction for the warped product manifold (see [4], [5] and [11]) is the same; for example considering the vertical vector fields U, V, as lift of vector fields of F, the development of the formulas remains the same. X, Y are lift from B, they are horizontal and so constant on fibers, then for example $V[X, Y] = 0$, and the inner product between a vector field on B with one on F is zero (i.e., $\langle X, V \rangle = 0$).

Practically, from the differential geometric point of view, a special type of Einstein sequential warped product manifold $(M, \bar{g}) = (B_1 \times B_2, g_1 + g_2) \times_f (\mathbb{R}^d, \ddot{g})$, is obtained, which allows to cover a wider variety of exact solutions of Einstein's field equation, without complicating the calculations much, compared to the Einstein warped-product manifolds with Ricci-flat fiber $(F; \ddot{g})$, see also [12]. From the derived geometric point of view, the outcome PNDP-manifold will be a manifold with "virtual" dimension $(n+(d-rank(E)))$.

PNDP-metric: Referring to a PNDP-manifolds, with negative "virtual" dimensional fiber, and for not confusing its metric with the metrics of a "classic" Einstein warped product manifold, the Riemannian or pseudo-Riemannian metric of the fiber-manifold are denoted with the following notation to indicate that F has negative "virtual" dimensions: $\ddot{g} = \Sigma(d\psi^i)^2_{(m)}$, where m is the negative "virtual" dimension of F.
So, the general metric form of a PNDP-manifold is:

$$\bar{g} = g - f^2(\Sigma_{i=1}^n (d\psi^i)^2)_{(m)} = (g_1 + g_2) + (f_1 + f_2)^2(\Sigma_{i=1}^n (d\psi^i)^2)_{(m)}. \tag{3.8}$$

Example 3.1. *Trivial Example - A type of flat PNDP-manifold with positive "virtual" dimension.*
The manifold $(\mathbb{R}^2 \times \mathbb{R}^2) \times [(\mathbb{R}^2 + E)]$ *(with* $rank(E) = 4$*) is a* $(4-2)$*-PNDP-manifold Ricci-flat. In fact it satisfies the system (3.4) for constant* $f = 1$ *(*f_1 *and* f_2 *both constants), we have* $n + m = n - d = 4 - 2 = 2 > 0$*, where* $rank(E) = -4$*, so* $m = -2$*. Thus we have to consider* $dim(B_1) = dim(F)$ *and its metric will be:* $ds^2 = dt^2 + dx^2 + dy^2 + dz^2 - (du^2 + dv^2)_{(-2)}$.

4. Interpretation of "virtual" dimensions

In this section we will deal with the interpretation of "virtual" dimensions, in order to show the speculative/applicative potential of PNDP-manifolds.
Since m is a "virtual" negative dimension ($m = -d$), the dimension of the PNDP-manifold (M, g_M) will be virtual too ($dim(M) = dim(B) + dim(F) = dim(B) + m = dim(B) - d$, where we have $dim(B) = dim(B_1) + dim(B_2)$, because $B = B_1 \times B_2$). Thus $dim(M)$ could be"virtual" positive, zero or negative. This means, from speculative point of view, that the negative "virtual" dimensions of F interact and "virtually" cancel each other with the positive dimensions of B.
For example, since $\bar{Ric} = \lambda\bar{g}$, from the definition of PNDP-manifold (Definition 3.1) we have that B_2-manifold is also Einstein with the same λ-constant of the PNDP-manifold, i.e., $Ric_2 = \lambda g_2$, and when "virtual" $dim(M)_{\mathcal{V}} > 0$, then (again for Definition 3.1), $dim(M)_{\mathcal{V}} = dim(B_2)$; in this way from a speculative/applicative point of view, will be

considered a special projection that acts as desuspension, such that it "projects" M into B_2, which the latter has real dimensions, not virtual ones, and this is interpreted in the following way: the manifold M "emerges" as the manifold B_2 (we remember that both (M and B_2) are Einstein, with the same λ-constant and in this case $dim(M)_{\mathcal{V}} = dim(B_2)$), the rest of M is hidden, and this is in favor of the interpretation that the negative "virtual" dimensions of F interact and cancel each other out with the positive dimension of B_1.
Considerations will also be made on $dim(M)_{\mathcal{V}} = 0$ and $dim(M)_{\mathcal{V}} < 0$. Below are the three types of PNDP-manifolds considered:

Type I) the $(n, -n)$-PNDP manifold that has overall, zero "virtual" dimension ($dim(M) = dim(B) + dim(F) = (n + (-n)) = 0$). The speculative result may be interpreted as an "invisible" manifold, a "point-like manifold" (zero-dimension) with "hidden" dimensions,

Type II) the $(n, -d)$-PNDP manifold, where n (the dimension of the base-manifold B) is different from d (with d a positive integer number that is the dimension of the underlying manifold $\mathbb{R}^d$ of the fiber-manifold F) such that $dim = (n + (-d)) > 0$. The particular speculative feature of this manifold is that it appears as another Einstein-manifold (i.e., B_2 manifold), and

Type III) it is like the *Type II*, but $dim = (n + (-d)) < 0$. It has the speculative feature of being considered, through special projection, like $||(n - d)||$-th desuspension of a point.

Definition 4.1. *In general, given an n-dimensional space X, the suspension ΣX has dimension $n + 1$. Thus, the operation of suspension creates a way of moving up in dimension. The inverse operation Σ^{-1}, is called desuspension. Therefore, given an n-dimensional space X, the desuspension $\Sigma^{-1}X$ has dimension $n - 1$, (see [13]).*

As mentioned above, since the interpretation wants that each negative "virtual" dimension acts on a positive dimension canceling each other, we consider the relation between the "virtual" dimension of a PNDP-manifold and the usual geometric concept of "dimension", as a desuspension interpreted by a special projection.
We have $B = (B_1 \times B_2)$ and $F := (\mathbb{R}^d, E, S)$, then PNDP$= B \times_f F = (B_1 \times B_2) \times_{(f_1+f_2)} F$, with $dim(B) - dim(\mathbb{R}^d) = dim(\text{PNDP}) = (n - d)$, therefore:

Type I) if $(n - d) = 0$, (i.e., system solutions (3.3)), we have the projection:

$$\pi_{(0)} : \text{PNDP} \to \text{point-like manifold},$$

Type II) if $(n-d) > 0$ (i.e. system solutions (3.4)) we have the projection:

$$\pi_{(>0)} : \text{PNDP} \to B_2,$$

Type III) if $(n-d) < 0$, (i.e., system solutions (3.3)), we have the projection:

$$\pi_{(<0)} : \text{PNDP} \to \Sigma^{n-d}(p),$$

with $\Sigma^{n-d}(p)$, the $||(n-d)||$-th desuspension of a point.

Example 4.1. *Speculative Example - Let* $(B_1 \times B_2) \times_f F$ *be a* $(8-4)$*-PNDP-manifold with* f *non-constant, and since* $n+m = n-d = 8-4 = 4 > 0$*, then (from Defintion 4)* $dim(B_1) = -dim(F)$*, so* $dim(B_1) = dim(\mathbb{R}^4)$*. Therefore the PNDP-manifold will satisfy the system (3.4) and* B_2 *will be an Einstein-manifold, i.e.,* $Ric_2 = \lambda g_2$*, then we will have:*
$\pi_4 : (8-4)$*-PNDP* $\to B_2$*, where* $rank(E) = -8$*. Hence with this interpretation of the "virtual" dimensions, from speculative point of view, the* $(8-4)$*-PNDP manifold is identified with the Einstein-manifold* B_2*.*

Example 4.2. *Speculative Example - Let* $(B_1 \times B_2) \times_f F$ *be a* $(6-6)$*-PNDP-manifold with* f *non-constant, and since* $n+m = n-d = 6-6 = 0$*, then* $dim(B) = -dim(F)$*. Therefore our PNDP-manifold will satisfy the system (3.3), and we will have:*
$\pi_0 : (6-6) - PNDP \to$ *point-like manifold (zero-dimension), where* $rank(E) = -12$.

Example 4.3. *Speculative Example - Let* $(B_1 \times B_2) \times_f F$ *be a* $(6-8)$*-PNDP-manifold with* f *non-constant, and since* $n+m = n-d = 6-8 = -2 < 0$*. Therefore our PNDP-manifold will satisfy again the system (3.3), and we will have:*
$\pi_{-2} : (6-8) - PNDP \to \Sigma^{-2}(p)$ *(double desuspension of a point, i.e.,* -2*-dimensional manifold), where* $rank(E) = -16$.

Example 4.4. *Speculative Example using Trivial Example - Considering the manifold of the Trivial Example, where we have the special case* $B_1 = B_2$*, that is* $\mathbb{R}^2$*, the desuspension/projection will be:* $\pi_2 : (\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 + E) \to \mathbb{R}^2$*, i.e., we identify the* $(4-2)$*-PNDP manifold with* $\mathbb{R}^2$*.*

5. Possible Multi-Disciplinary Applications

Since the dimensions of a PNDP-manifold is not related with the usual geometric concept of dimension (we consider them as "virtual" dimensions), from a speculative point of view a PNDP-manifold can consider a concept of virtual dimensions referring to how things are observed. In fact, having chosen a manifold and defining a certain

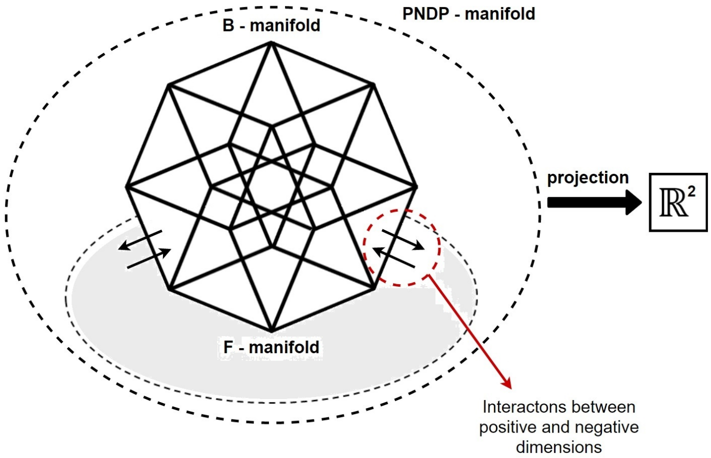

FIGURE 1. *The figure represents the Trivial Example according to the speculative approach to "virtual" dimensions described in the chapter. The PNDP-maifold is* $(\mathbb{R}^2 \times \mathbb{R}^2) \times [(\mathbb{R}^2 + E)]$ *(where the base-manifld is represented in the figure as a tesseratto) with* $rank(E) = -4$*, and following the approach about the interaction between dimensions (in which the negative "virtual" dimensions of* F *interact canceling/hiding with the dimensions of* B_1*), the manifold resulting from the projection will be* $\mathbb{R}^2$*.*

algebra on it, the manifold in respect to that algebra will admit a dimension that we call "virtual", because if we don't look at that algebra, but we look only at the underlying manifold, the value of its dimension will be different. So, in a PNDP-manifold, depending on what we consider, we get different dimensional results. So, from speculative point of view we want suggest that nature manifest itself in a certain way not necessarily as we observe it. In this respect, PNDP-manifold could be used as a tool to reveal the "emergent" dimensional aspect of nature, i.e., the "emergent space", and with a correct interpretation it is possible to consider a new type of "hidden" dimensions. In this perspective, the PNPD-manifold, that we have described in detail in the previous sections (sections 3 and 4), can be considered in many types of applications, as it was done in [14], [15] and [16], where the authors considered speculatively the string theory, D-branes and the discrete gravity theory.

This new point of view, starts from the assumption that some dimensions of space-time, by their intrinsic nature or by some initial situation, behave in such a way as to be able to mathematically describe them as "virtual" negative and this, within the configuration, implies interactions with the other dimensions that are present. In fact, the space could be a secondary property created by other more fundamental forces, and in that sense, dimensions could also vanish, because, for example, non-gravitational extra dimensions can be dynamically generated by fundamentally four-dimensional gauge

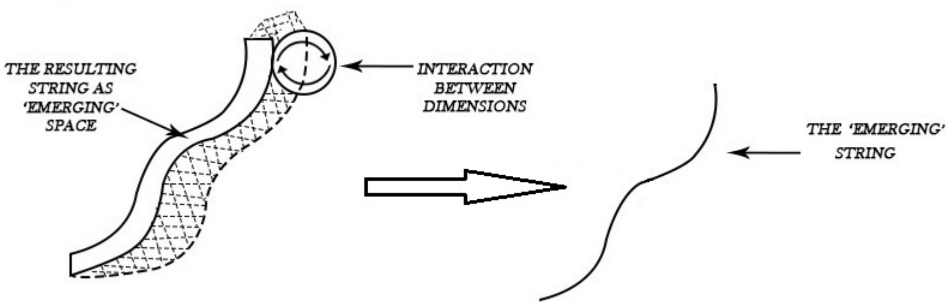

Figure 2. *A* $2-1$*- PNDP is shown,.e.* $(I_1 \times I_2) \times (I_3 + E)$*. From the interaction between the positive and virtual negative dimensions, a line interval emerges, topologically equivalent to a string.*

theories. The negative "virtual" dimension therefore corresponds to a possible mathematical description of a dimension in which one type of particular fundamental forces allow other dimensions to "emerge".

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