A BOUNDARY VALUE PROBLEM FOR SYSTEM OF DIFFERENTIAL EQUATIONS WITH PIECEWISE-CONSTANT ARGUMENT OF GENERALIZED TYPE

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1. Introduction

Mathematical modeling of processes with discontinuity effects has necessitated the need to develop the theory of differential equations with discontinuities. An important class of such equations is comprised of differential equations with piecewise constant argument (DEPCA). The study of DEPCA was initiated by Busenberg, Cooke, Shah, and Wiener [1, 2, 3].

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[1] **Busenberg S., Cooke K.L.** Models of vertically transmitted diseases with sequential-continuous dynamics, *in "Nonlinear Phenomena in Mathematical Sciences"(V. Lakshmikantham, Ed.)*, pp. 179-187, Academic Press, NY, 1982.,

[2] **Shah S. M., Wiener J.** Advanced differential equations with piecewise constant argument deviations, *Intern. J. Math. and Math. Sciences*, 6 (1983), 671-703.,

[3] **Cooke K.L., Wiener J.** Retarded differential equations with piecewise constant delays, *J. Math. Anal. Appl.*, 99 (1984), 265-297.

The questions of the existence and uniqueness of solutions to DEPCA, their oscillations and stability, integral manifolds and periodic solutions have been extensively discussed by many authors (see References).

When modeling with DEPCA, the deviation of the argument, taken as the greatest integer function, is always constant and equal to one. But this approach can contradict real phenomena. The generalization of DEPCA has been undertaken by Akhmet [4, 5, 6, 7].

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[4] **Akhmet M.U.** Integral manifolds of differential equations with piecewise constant argument of generalized type, *Nonl. Anal.*, 66 (2007), 367-383.,

[5] **Akhmet M.U.** Almost periodic solutions of differential equations with piecewise constant argument of generalized type, *J. Math. Anal. Appl.*, 336 (2007), 646-663.,

[6] **Akhmet M.U.** Almost periodic solutions of differential equations with piecewise constant argument of generalized type, *Nonl. Anal.: Hybrid Systems and Appl.*, 2 (2008), 456-467.,

[7] **Akhmet M.U.** Stability of differential equations with piecewise constant argument of generalized type, *Nonl. Anal.: Theory, Methods and Appl.*, 68 (2008), 794-803.

In his works the greatest integer function as deviating argument was replaced by an arbitrary piecewise constant function. Thus, differential equations with piecewise constant argument of generalized type (DEPCAG) are more suitable for modeling and solving various application problems, including areas of neural networks, discontinuous dynamical systems, hybrid systems, etc. To date, the theory of DEPCAG on the entire axis has been developed and their applications have been implemented. The results have been extended to periodic impulse systems of DEPCAG (see References).

Along with the study of various properties of DEPCA, a number of authors investigated the questions of solvability and construction of solutions to boundary value problems for these equations on a finite interval (see References).
For DEPCAG, however, the questions of solvability of boundary value problems on a finite interval still remain open.
This issue can be resolved by developing constructive methods.

2. Statement of problem

On $[0, T]$, we consider the following two-point boundary value problem for a system of DEPCAG:

$$\frac{\mathbf{dx}}{\mathbf{dt}} = \mathbf{A(t)x + A_0(t)x(\gamma(t)) + f(t)}, \qquad \mathbf{x} \in \mathbb{R}^{\mathbf{n}}, \quad \mathbf{t} \in (\mathbf{0}, \mathbf{T}), \tag{1}$$

$$\mathbf{Bx(0) + Cx(T) = d}, \qquad \mathbf{d} \in \mathbb{R}^{\mathbf{n}}. \tag{2}$$

Here $\mathbf{x(t) = col(x_1(t), x_2(t), ..., x_n(t))}$ is the unknown function, $(\mathbf{n \times n})$ matrices $\mathbf{A(t)}$, $\mathbf{A_0(t)}$ and $\mathbf{n}$-vector $\mathbf{f(t)}$ are continuous on $[\mathbf{0}, \mathbf{T}]$;

$\gamma(t) = \zeta_j$ if $\mathbf{t} \in [\theta_{\mathbf{j}}, \theta_{\mathbf{j+1}})$, $\mathbf{j} = \overline{\mathbf{0, N-1}}$;

$\theta_{\mathbf{j}} \le \zeta_{\mathbf{j}} \le \theta_{\mathbf{j+1}}$ for all $\mathbf{j = 0, 1, \ldots, N-1}$;

$\mathbf{0} = \theta_{\mathbf{0}} < \theta_{\mathbf{1}} < \ldots < \theta_{\mathbf{N-1}} < \theta_{\mathbf{N}} = \mathbf{T}$;

$\mathbf{B}$ and $\mathbf{C}$ are constant $(\mathbf{n \times n})$ matrices, and $\mathbf{d}$ is constant vector.

A function $\mathbf{x}^*(\mathbf{t}) : [\mathbf{0}, \mathbf{T}] \to \mathbb{R}^{\mathbf{n}}$ is a solution to problem (1), (2) if:

(i) $\mathbf{x}^*(\mathbf{t})$ is continuous on $[\mathbf{0}, \mathbf{T}]$;

(ii) $\mathbf{x}^*(\mathbf{t})$ is differentiable on $[\mathbf{0}, \mathbf{T}]$ with the possible exception of the points $\theta_{\mathbf{j}}$, $\mathbf{j} = \overline{\mathbf{0}, \mathbf{N}-\mathbf{1}}$, at which the one-sided derivatives exist;

(iii) $\mathbf{x}^*(\mathbf{t})$ satisfies the system of equations (1) on each interval $(\theta_{\mathbf{j}}, \theta_{\mathbf{j+1}})$, $\mathbf{j} = \overline{\mathbf{0}, \mathbf{N}-\mathbf{1}}$; at the points $\theta_{\mathbf{j}}$, $\mathbf{j} = \overline{\mathbf{0}, \mathbf{N}-\mathbf{1}}$, system (1) is satisfied by the right-hand derivative of $\mathbf{x}^*(\mathbf{t})$;

(iv) $\mathbf{x}^*(\mathbf{t})$ satisfies the boundary condition (2) at $\mathbf{t} = \mathbf{0}$ and $\mathbf{t} = \mathbf{T}$.

The aim of report is to develop a constructive method for investigation and solving the boundary value problem (1), (2), as well as to construct algorithms for finding approximate solutions to the problem.

To this end, we use the parametrization method [8] and a new approach to the concept of the general solution.
This approach was originally introduced for linear Fredholm IDEs [9] and then applied to families of loaded DEs [10] and to ODEs [11]

[8] **Dzhumabaev D.S.** On one approach to solve the linear boundary value problems for Fredholm integro-differential equations, *J. Comput. Appl. Math.*, 294 (2016), 342-357.

[9] **Dzhumabaev D.S.** New general solutions to linear Fredholm integro-differential equations and their applications on solving the BVPs, *J. Comput. Appl. Math.*, 327 (2018), 79-108.

[10] **Dzhumabaev D.S.** Well-posedness of nonlocal boundary-value problem for a system of loaded hyperbolic equations and an algorithm for finding its solution, *J. Math. Anal. Appl.*, 461 (2018), 817-836.

[11] **Dzhumabaev D.S.** New general solutions of ordinary differential equations and the methods for the solution of boundary - value problems, *Ukranian Math. J.*, 71 (2019), 1006-1031.

3. Scheme of the method

Let $\mathbf{\Delta_N}$ denote the partition of the interval $[\mathbf{0}, \mathbf{T})$ by points

$$\mathbf{t} = \theta_\mathbf{r}, \quad \mathbf{r} = \overline{\mathbf{1}, \mathbf{N} - \mathbf{1}}: \quad [\mathbf{0}, \mathbf{T}) = \bigcup_{\mathbf{r}=\mathbf{1}}^{\mathbf{N}} [\theta_{\mathbf{r}-\mathbf{1}}, \theta_\mathbf{r}).$$

We define the following spaces:
$\mathbf{C}([\mathbf{0}, \mathbf{T}], \mathbb{R}^\mathbf{n})$ is the space of continuous functions $\mathbf{x} : [\mathbf{0}, \mathbf{T}] \to \mathbb{R}^\mathbf{n}$ with the norm

$$\|\mathbf{x}\|_\mathbf{1} = \max_{\mathbf{t} \in [\mathbf{0}, \mathbf{T}]} ||\mathbf{x}(\mathbf{t})|| = \max_{\mathbf{t} \in [\mathbf{0}, \mathbf{T}]} \max_{\mathbf{i} = \overline{\mathbf{1}, \mathbf{n}}} |\mathbf{x_i}(\mathbf{t})|;$$

$\mathbf{C}([\mathbf{0}, \mathbf{T}], \mathbf{\Delta_N}, \mathbb{R}^{\mathbf{nN}})$ is the space of functions systems $\mathbf{x}[\mathbf{t}] = (\mathbf{x_1}(\mathbf{t}), \mathbf{x_2}(\mathbf{t}), \ldots, \mathbf{x_N}(\mathbf{t}))$, where $\mathbf{x_r} : [\theta_{\mathbf{r}-\mathbf{1}}, \theta_\mathbf{r}) \to \mathbb{R}^\mathbf{n}$ are continuous functions that have finite left-hand limits $\lim_{\mathbf{t} \to \theta_\mathbf{r} - \mathbf{0}} \mathbf{x_r}(\mathbf{t})$ for all $\mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}$, with the norm

$$\|\mathbf{x}[\cdot]\|_\mathbf{2} = \max_{\mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}} \sup_{\mathbf{t} \in [\theta_{\mathbf{r}-\mathbf{1}}, \theta_\mathbf{r})} |\mathbf{x_r}(\mathbf{t})|.$$

Let $\mathbf{x_r(t)}$ denote the restriction of the function $\mathbf{x(t)}$ to the $\mathbf{r}$th subinterval $[\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}})$, i.e.

$$\mathbf{x_r(t) = x(t)} \quad \text{for} \quad \mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}}), \quad \mathbf{r} = \overline{\mathbf{1, N}}.$$

Then the system $\mathbf{x[t] = (x_1(t), x_2(t), \ldots, x_N(t))}$ belongs to the space $\mathbf{C([0,T], \Delta_N}, \mathbb{R}^{\mathbf{nN}})$, and its elements $\mathbf{x_r(t)}$, $\mathbf{r} = \overline{\mathbf{1, N}}$, satisfy the following system of DEPCAG

$$\frac{\mathbf{dx_r}}{\mathbf{dt}} = \mathbf{A(t)x_r(t) + A_0(t)x_r}(\zeta_{\mathbf{r-1}}) + \mathbf{f(t)}, \tag{3}$$

$$\mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}}), \qquad \mathbf{r} = \overline{\mathbf{1, N}}.$$

Here we take into account that

$$\gamma(\mathbf{t}) = \zeta_{\mathbf{j}} \qquad \text{if} \qquad \mathbf{t} \in [\theta_{\mathbf{j}}, \theta_{\mathbf{j+1}}), \qquad \mathbf{j} = \overline{\mathbf{0, N-1}}.$$

We introduce parameters $\lambda_r = x_r(\zeta_{r-1})$, $r = \overline{1, N}$. By making the substitution

$z_r(t) = x_r(t) - \lambda_r$ on each subinterval $[\theta_{r-1}, \theta_r)$,

we get the system of differential equations with parameters

$$\frac{dz_r}{dt} = A(t)(z_r(t) + \lambda_r) + A_0(t)\lambda_r + f(t), \qquad (4)$$

$$t \in [\theta_{r-1}, \theta_r), \qquad r = \overline{1, N},$$

subject to the initial conditions

$$z_r(\zeta_{r-1}) = 0, \qquad r = \overline{1, N}. \qquad (5)$$

Thus we obtain the Cauchy problems (4), (5) for systems of ODEs with parameters on the subintervals $[\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$.

For fixed $\lambda_r \in \mathbb{R}^n$ and r, the Cauchy problem (4), (5) has a unique solution $z_r(t, \lambda_r)$, and the system $z[t, \lambda] = (z_1(t, \lambda_1), \ldots, z_N(t, \lambda_N))$ belongs to $C([0, T], \Delta_N, \mathbb{R}^{nN})$.

A system of functions

$$\mathbf{z}[\mathbf{t}, \lambda] = (\mathbf{z_1}(\mathbf{t}, \lambda_\mathbf{1}), \ldots, \mathbf{z_N}(\mathbf{t}, \lambda_\mathbf{N}))$$

is called a solution to the Cauchy problems with parameters (4), (5).

If the system of functions $\widetilde{\mathbf{x}}[\mathbf{t}] = (\widetilde{\mathbf{x}}_\mathbf{1}(\mathbf{t}), \widetilde{\mathbf{x}}_\mathbf{2}(\mathbf{t}), \ldots, \widetilde{\mathbf{x}}_\mathbf{N}(\mathbf{t}))$ belongs to $\mathbf{C}([\mathbf{0}, \mathbf{T}], \mathbf{\Delta_N}, \mathbb{R}^{\mathbf{nN}})$, and the functions $\widetilde{\mathbf{x}}_\mathbf{r}(\mathbf{t})$, $r = \overline{1, N}$, satisfy equations (3), then the system of functions

$\mathbf{z}[\mathbf{t}, \widetilde{\lambda}] = (\mathbf{z_1}(\mathbf{t}, \widetilde{\lambda}_\mathbf{1}), \mathbf{z_2}(\mathbf{t}, \widetilde{\lambda}_\mathbf{2}), \ldots, \mathbf{z_N}(\mathbf{t}, \widetilde{\lambda}_\mathbf{N}))$ with elements $\mathbf{z_r}(\mathbf{t}, \widetilde{\lambda}_\mathbf{r}) = \widetilde{\mathbf{x}}_\mathbf{r}(\mathbf{t}) - \widetilde{\lambda}_\mathbf{r}$, $\widetilde{\lambda}_\mathbf{r} = \widetilde{\mathbf{x}}_\mathbf{r}(\zeta_{\mathbf{r-1}}), \quad \mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}$,

is a solution to the Cauchy problem with parameters (4), (5) with $\lambda_\mathbf{r} = \widetilde{\lambda}_\mathbf{r}$.

Conversely, if a system of functions

$$\mathbf{z}[\mathbf{t}, \lambda^*] = (\mathbf{z_1}(\mathbf{t}, \lambda^*_\mathbf{1}), \mathbf{z_2}(\mathbf{t}, \lambda^*_\mathbf{2}), \ldots, \mathbf{z_N}(\mathbf{t}, \lambda^*_\mathbf{N}))$$

is a solution to problem (4), (5) with $\lambda_\mathbf{r} = \lambda^*_\mathbf{r}$, $\mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}$, then system of functions $\mathbf{x}^*[\mathbf{t}] = (\mathbf{x}^*_\mathbf{1}(\mathbf{t}), \mathbf{x}^*_\mathbf{2}(\mathbf{t}), \ldots, \mathbf{x}^*_\mathbf{N}(\mathbf{t}))$ with $\mathbf{x}^*_\mathbf{r}(\mathbf{t}) = \lambda^*_\mathbf{r} + \mathbf{z_r}(\mathbf{t}, \lambda^*_\mathbf{r})$, $\mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}$, belongs to $\mathbf{C}([\mathbf{0}, \mathbf{T}], \mathbf{\Delta_N}, \mathbb{R}^{\mathbf{nN}})$ and the functions $\mathbf{x}^*_\mathbf{r}(\mathbf{t})$, $\mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}$, satisfy equations (3).

4. The concept of new general solution to system (1)

We use the new concept of the general solution proposed in [9] to introduce the new general solution to the system of DEPCAG (1).

Definition 1.

Let $\mathbf{z}[\mathbf{t},\lambda] = (\mathbf{z_1}(\mathbf{t},\lambda_\mathbf{1}), \mathbf{z_2}(\mathbf{t},\lambda_\mathbf{2}), \ldots, \mathbf{z_N}(\mathbf{t},\lambda_\mathbf{N}))$ *be a solution of the Cauchy problem (4),(5) with parameters* $\lambda = (\lambda_\mathbf{1}, \lambda_\mathbf{2}, ..., \lambda_\mathbf{N}) \in \mathbb{R}^{\mathbf{nN}}$. *Then the function* $\mathbf{x}(\mathbf{\Delta_N}, \mathbf{t}, \lambda)$, *defined by the equalities*

$\mathbf{x}(\mathbf{\Delta_N}, \mathbf{t}, \lambda) = \lambda_\mathbf{r} + \mathbf{z_r}(\mathbf{t}, \lambda_\mathbf{r})$ *for* $\mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_\mathbf{r}]$, $\mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}$, *and*

$\mathbf{x}(\mathbf{\Delta_N}, \mathbf{T}, \lambda) = \lambda_\mathbf{N} + \lim\limits_{\mathbf{t} \to \mathbf{T}-\mathbf{0}} \mathbf{z_N}(\mathbf{t}, \lambda_\mathbf{N})$,

is called the $\mathbf{\Delta_N}$*-general solution to the system of DEPCAG (1).*

It follows from Definition 1 that the $\mathbf{\Delta_N}$-general solution depends on $\mathbf{N}$ arbitrary vectors $\lambda_\mathbf{r} \in \mathbb{R}^\mathbf{n}$ and satisfies the system of DEPCAG (1) for all $\mathbf{t} \in (\mathbf{0}, \mathbf{T}) \setminus \{\theta_\mathbf{p}, \mathbf{p} = \overline{\mathbf{1}, \mathbf{N}-\mathbf{1}}\}$.

Let $\mathbf{X_r(t)}$ be a fundamental matrix of the system of ODEs

$$\frac{\mathbf{dz_r}}{\mathbf{dt}} = \mathbf{A(t)z_r(t)}, \qquad \mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}}], \qquad \mathbf{r} = \overline{\mathbf{1, N}}.$$

Hence, the solutions to the Cauchy problems with parameters (4),(5) can be represented as

$$\mathbf{z_r(t)} = \mathbf{X_r(t)} \int_{\zeta_{\mathbf{r-1}}}^{\mathbf{t}} \mathbf{X_r^{-1}}(\tau)[\mathbf{A}(\tau) + \mathbf{A_0}(\tau)]\mathbf{d}\tau\lambda_{\mathbf{r}} +$$

$$+\mathbf{X_r(t)} \int_{\zeta_{\mathbf{r-1}}}^{\mathbf{t}} \mathbf{X_r^{-1}}(\tau)\mathbf{f}(\tau)\mathbf{d}\tau,$$

$$\mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}}], \qquad \mathbf{r} = \overline{\mathbf{1, N}}.$$

We consider the Cauchy problems on the partition subintervals

$$\frac{\mathbf{dx}}{\mathbf{dt}} = \mathbf{A}(\mathbf{t})\mathbf{x} + \mathbf{P}(\mathbf{t}), \qquad \mathbf{x}(\zeta_{\mathbf{r-1}}) = \mathbf{0}, \qquad \mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}}], \quad (6)$$

$$\mathbf{r} = \overline{\mathbf{1}, \mathbf{N}},$$

where $\mathbf{P}(\mathbf{t})$ is a square matrix of order $\mathbf{n}$ or an $\mathbf{n}$-dimensional vector, continuous on $[\mathbf{0}, \mathbf{T}]$, $\theta_{\mathbf{r-1}} \leq \zeta_{\mathbf{r-1}} \leq \theta_{\mathbf{r}}$ for all $\mathbf{r} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{N}$.

Let $\mathbf{A}_{\mathbf{r}}(\mathbf{P}, \mathbf{t})$ denote the unique solution of the Cauchy problem (6) on each $\mathbf{r}$th subinterval. It follows from the unique solvability of the Cauchy problem for linear ODEs that

$$\mathbf{A}_{\mathbf{r}}(\mathbf{P}, \mathbf{t}) = \mathbf{X}_{\mathbf{r}}(\mathbf{t}) \int\limits_{\zeta_{\mathbf{r-1}}}^{\mathbf{t}} \mathbf{X}_{\mathbf{r}}^{-\mathbf{1}}(\tau)\mathbf{P}(\tau)\mathbf{d}\tau, \qquad \mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}}],$$

$$\mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}.$$

We can now represent the $\mathbf{\Delta_N}$-general solution of system of DEPCAG (1) in the form

$$\mathbf{x}(\mathbf{\Delta_N}, \mathbf{t}, \lambda) = \lambda_\mathbf{r} + \mathbf{A_p}(\mathbf{A} + \mathbf{A_0}, \mathbf{t})\lambda_\mathbf{r} + \mathbf{A_p}(\mathbf{f}, \mathbf{t}), \qquad (7)$$

$$\mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_\mathbf{r}], \qquad \mathbf{r} = \overline{\mathbf{1}, \mathbf{N} - \mathbf{1}},$$

$$\mathbf{x}(\mathbf{\Delta_N}, \mathbf{t}, \lambda) = \lambda_\mathbf{N} + \mathbf{A_N}(\mathbf{A} + \mathbf{A_0}, \mathbf{t})\lambda_\mathbf{N} + \mathbf{A_N}(\mathbf{f}, \mathbf{t}), \qquad (8)$$

$$\mathbf{t} \in [\theta_{\mathbf{N-1}}, \theta_\mathbf{N}].$$

The following statement justifies the fact that the function $\mathbf{x}(\mathbf{\Delta_N}, \mathbf{t}, \lambda)$ can be considered as the general solution of system (1).

Theorem 1.

Let $\widetilde{\mathbf{x}}(\mathbf{t})$ *be a pointwise continuous on* $[\mathbf{0}, \mathbf{T}]$ *function with possible discontinuity points* $\mathbf{t} = \theta_{\mathbf{p}}$, $\mathbf{p} = \overline{\mathbf{1}, \mathbf{N}-\mathbf{1}}$,

and let $\mathbf{x}(\mathbf{\Delta_N}, \mathbf{t}, \lambda)$ *be the* $\mathbf{\Delta_N}$*-general solution of the system of DEPCAG (1).*

Suppose that the function $\widetilde{\mathbf{x}}(\mathbf{t})$ *has a continuous derivative and satisfies equation (1) for all* $\mathbf{t} \in (\mathbf{0}, \mathbf{T}) \backslash \{\theta_{\mathbf{p}}, \mathbf{p} = \overline{\mathbf{1}, \mathbf{N}-\mathbf{1}}\}$.

Then there exists a unique $\widetilde{\lambda} = (\widetilde{\lambda}_{\mathbf{1}}, \widetilde{\lambda}_{\mathbf{2}}, ..., \widetilde{\lambda}_{\mathbf{N}}) \in \mathbb{R}^{\mathbf{nN}}$ *such that the equality*

$\mathbf{x}(\mathbf{\Delta_N}, \mathbf{t}, \widetilde{\lambda}) = \widetilde{\mathbf{x}}(\mathbf{t})$ *holds for all* $\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$.

Lemma 1.

Let $\mathbf{x}^*(\mathbf{t})$ *be a solution to the system of DEPCAG (1), and let* $\mathbf{x}(\mathbf{\Delta_N}, \mathbf{t}, \lambda)$ *be the* $\mathbf{\Delta_N}$*-general solution to equation (1). Then there exists a unique* $\lambda^* = (\lambda^*_{\mathbf{1}}, \ldots, \lambda^*_{\mathbf{N}}) \in \mathbb{R}^{\mathbf{nN}}$ *such that* $\mathbf{x}(\mathbf{\Delta_N}, \mathbf{t}, \lambda^*) = \mathbf{x}^*(\mathbf{t})$ *for all* $\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$.

If $\mathbf{x}(\mathbf{t})$ is a solution to system (1) and $\mathbf{x}[\mathbf{t}] = (\mathbf{x_1}(\mathbf{t}), \mathbf{x_2}(\mathbf{t}), \ldots, \mathbf{x_N}(\mathbf{t}))$ is the system of functions composed of its restrictions to the subintervals $[\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}}]$, $\mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}$, then the following equalities hold:

$$\lim_{\mathbf{t} \to \theta_{\mathbf{p}} - \mathbf{0}} \mathbf{x_p}(\mathbf{t}) = \mathbf{x_{p+1}}(\theta_{\mathbf{p}}), \qquad \mathbf{p} = \overline{\mathbf{1}, \mathbf{N} - \mathbf{1}}. \tag{9}$$

These equations express the conditions for the continuity of the solution to system (1) at the interior points of the partition $\mathbf{\Delta_N}$.

Theorem 2.

Let a system of functions $\mathbf{x[t]} = (\mathbf{x_1(t)}, \mathbf{x_2(t)}, \ldots, \mathbf{x_N(t)})$ *belong to the space* $\mathbf{C([0,T], \Delta_N, \mathbb{R}^{nN})}$.

Suppose that the functions $\mathbf{x_r(t)}$, $\mathbf{r} = \overline{\mathbf{1,N}}$, *satisfy the systems of equations (3) and the continuity conditions (9).*

Then the function $\mathbf{x^*(t)}$, *defined by the equalities*

$$\mathbf{x^*(t)} = \mathbf{x_r(t)}, \qquad \mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}}), \qquad \mathbf{r} = \overline{\mathbf{1,N}},$$

and

$$\mathbf{x^*(T)} = \lim_{\mathbf{t} \to \mathbf{T-0}} \mathbf{x_N(t)},$$

is continuous on $[\mathbf{0,T}]$, *continuously differentiable on* $(\mathbf{0,T})$ *and satisfies the system of DEPCAG (1).*

5. Solvability of problem (1), (2)

The introduction of the $\mathbf{\Delta_N}$-general solution allows one to reduce the solvability of the boundary value problem under consideration to that of a system of linear algebraic equations in arbitrary vectors $\lambda_\mathbf{r} \in \mathbb{R}^\mathbf{n}$, $\mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}$.

By substituting the expressions (7), (8) of the $\mathbf{\Delta_N}$-general solution into the boundary condition (2) and the continuity conditions (9), we obtain the system of linear algebraic equations

$$\mathbf{B}\lambda_\mathbf{1} + \mathbf{B}\mathbf{A}_\mathbf{1}(\mathbf{A} + \mathbf{A}_\mathbf{0}, \theta_\mathbf{0})\lambda_\mathbf{1} + \mathbf{C}\lambda_\mathbf{N} + \mathbf{C}\mathbf{A}_\mathbf{N}(\mathbf{A} + \mathbf{A}_\mathbf{0}, \mathbf{T})\lambda_\mathbf{N} =$$

$$= \mathbf{d} - \mathbf{B}\mathbf{A}_\mathbf{1}(\mathbf{f}, \theta_\mathbf{0}) - \mathbf{C}\mathbf{A}_\mathbf{N}(\mathbf{f}, \mathbf{T}), \tag{10}$$

$$\lambda_\mathbf{p} + \mathbf{A}_\mathbf{p}(\mathbf{A}, \theta_\mathbf{p})\lambda_\mathbf{p} - \lambda_\mathbf{p+1} - \mathbf{A}_\mathbf{p+1}(\mathbf{A} + \mathbf{A}_\mathbf{0}, \theta_\mathbf{p})\lambda_\mathbf{p+1} =$$

$$= -\mathbf{A}_\mathbf{p}(\mathbf{f}, \theta_\mathbf{p}) + \mathbf{A}_\mathbf{p+1}(\mathbf{f}, \theta_\mathbf{p}), \quad \mathbf{p} = \overline{\mathbf{1}, \mathbf{N} - \mathbf{1}}. \tag{11}$$

Let $\mathbf{Q}_*(\mathbf{\Delta_N})$ denote the square matrix of order $\mathbf{nN}$ composed of coefficients of $\lambda_\mathbf{r}$, $\mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}$. We can now rewrite system (10), (11) in the form

$$\mathbf{Q}_*(\mathbf{\Delta_N})\lambda = -\mathbf{F}_*(\mathbf{\Delta_N}), \qquad \lambda \in \mathbf{R}^{\mathbf{nN}}, \tag{12}$$

where

$$\mathbf{F}_*(\mathbf{\Delta_N}) = \big(-\mathbf{d} + \mathbf{B}\mathbf{A_1}(\mathbf{f}, \theta_\mathbf{0}) + \mathbf{C}\mathbf{A_N}(\mathbf{f}, \mathbf{T}),$$
$$\mathbf{A_1}(\mathbf{f}, \theta_\mathbf{1}) - \mathbf{A_2}(\mathbf{f}, \theta_\mathbf{1}), \mathbf{A_2}(\mathbf{f}, \theta_\mathbf{2}) - \mathbf{A_3}(\mathbf{f}, \theta_\mathbf{2}),$$
$$\ldots, \mathbf{A_{N-1}}(\mathbf{f}, \theta_\mathbf{N-1}) - \mathbf{A_N}(\mathbf{f}, \theta_\mathbf{N-1})\big) \in \mathbb{R}^{\mathbf{nN}}.$$

Theorems 1 and 2 imply that for any partition $\mathbf{\Delta_N}$ the following statement holds true.

LEMMA 2.

If $\mathbf{x}^*(\mathbf{t})$ *is a solution to problem (1), (2), and*
$\lambda^*_{\mathbf{r}} = \mathbf{x}^*(\zeta_{\mathbf{r-1}}), \quad \mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}$,
then the vector $\lambda^* = (\lambda^*_{\mathbf{1}}, \lambda^*_{\mathbf{2}}, \ldots, \lambda^*_{\mathbf{N}}) \in \mathbb{R}^{\mathbf{nN}}$ *is a solution to system (12).*

Conversely, if $\tilde{\lambda} = (\tilde{\lambda}_{\mathbf{1}}, \ldots, \tilde{\lambda}_{\mathbf{N}}) \in \mathbb{R}^{\mathbf{nN}}$ *is a solution to (12) and* $\mathbf{z}[\mathbf{t}, \tilde{\lambda}] = (\mathbf{z_1}(\mathbf{t}, \tilde{\lambda}_{\mathbf{1}}), \ldots, \mathbf{z_N}(\mathbf{t}, \tilde{\lambda}_{\mathbf{N}}))$ *is the solution to the Cauchy problem (4), (5) with* $\tilde{\lambda} \in \mathbb{R}^{\mathbf{nN}}$,
then the function $\tilde{\mathbf{x}}(\mathbf{t})$ *defined by the equalities*

$$\tilde{\mathbf{x}}(\mathbf{t}) = \tilde{\lambda}_{\mathbf{r}} + \mathbf{z_r}(\mathbf{t}, \tilde{\lambda}_{\mathbf{r}}), \qquad \mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}}), \qquad \mathbf{r} = \overline{\mathbf{1}, \mathbf{N}},$$

and

$$\tilde{\mathbf{x}}(\mathbf{T}) = \tilde{\lambda}_{\mathbf{N}} + \lim_{\mathbf{t} \to \mathbf{T}-\mathbf{0}} \mathbf{z_N}(\mathbf{t}, \tilde{\lambda}_{\mathbf{N}}),$$

is the solution to problem (1), (2).

Definition 2.

The boundary value problem (1), (2) is called uniquely solvable if it has a unique solution for any pair $(\mathbf{f}(\mathbf{t}), \mathbf{d})$ *with* $\mathbf{f}(\mathbf{t}) \in \mathbf{C}([\mathbf{0}, \mathbf{T}], \mathbb{R}^{\mathbf{n}})$ *and* $\mathbf{d} \in \mathbb{R}^{\mathbf{n}}$*.*

Lemma 2 and well-known theorems of linear algebra imply the following statements.

Theorem 3.

The boundary value problem (1), (2) has a solution if and only if the vector $\mathbf{F}_*(\mathbf{\Delta_N})$ *is orthogonal to the kernel of the transposed matrix* $(\mathbf{Q}_*(\mathbf{\Delta_N}))'$, *i.e. the equality*

$$(\mathbf{F}_*(\mathbf{\Delta_N}), \eta) = \mathbf{0}$$

holds for all $\eta \in \mathbf{Ker}(\mathbf{Q}_*(\mathbf{\Delta_N}))'$, *where* $(\cdot, \cdot)$ *is the dot product in* $\mathbb{R}^{\mathbf{nN}}$.

Theorem 4.

The boundary value problem (1), (2) is uniquely solvable if and only if the matrix $\mathbf{Q}_*(\mathbf{\Delta_N})$ *is invertible.*

6. Algorithm for finding solution of problem (1), (2)

Based on the results obtained in Section 5, we propose ***Algorithm A*** for solving the linear boundary value problem (1), (2).

Step 1 On the partition subintervals $[\theta_{r-1}, \theta_r]$, $r = \overline{1, N}$, solve the Cauchy problems

$$\frac{dz}{dt} = A(t)z + A(t) + A_0(t), \qquad z(\zeta_{r-1}) = 0,$$

and

$$\frac{dz}{dt} = A(t)z + f(t), \qquad z(\zeta_{r-1}) = 0,$$

to find the functions $A_r(A + A_0, \theta_r)$ and $A_r(f, \theta_r)$, respectively.

Here $\zeta_{r-1} \in [\theta_{r-1}, \theta_r]$, $r = \overline{1, N}$.

Step 2 Construct the system of linear algebraic equations (12) using the matrices and vectors found in Step 1.

Step 3 Find the solution $\lambda^* = (\lambda^*_{\mathbf{1}}, \lambda^*_{\mathbf{2}}, \ldots, \lambda^*_{\mathbf{N}}) \in \mathbb{R}^{\mathbf{nN}}$ to the system constructed in Step 2.
Note that the components of λ^* are the values of the solution to problem (1), (2) at the points $\zeta_{\mathbf{r-1}}$ of the partition subintervals: $\lambda^*_{\mathbf{r}} = \mathbf{x}^*(\zeta_{\mathbf{r-1}})$, $\mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}$.

Step 4 Find the values of the solution $\mathbf{x}^*(\mathbf{t})$ at the remaining points of the subintervals by solving the Cauchy problems

$$\frac{\mathbf{dz}}{\mathbf{dt}} = \mathbf{A}(\mathbf{t})\mathbf{z} + \mathbf{f}(\mathbf{t}), \qquad \mathbf{z}(\zeta_{\mathbf{r-1}}) = \lambda^*_{\mathbf{r}}, \qquad \mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}}).$$

The accuracy of the proposed algorithm depends on that of calculating the coefficients and right-hand parts of system (12).

7. Two-point boundary value problem for second order DEPCAG

Particular attention was paid to periodic, two-point and multipoint BVPs for second order DEPCA due to their wide application in natural sciences and engineering (see References).

On $[\mathbf{0}, \mathbf{T}]$ consider two-point BVP for second order DEPCAG

$$\ddot{\mathbf{u}} = \mathbf{a_1}(\mathbf{t})\dot{\mathbf{u}}(\mathbf{t}) + \mathbf{a_2}(\mathbf{t})\mathbf{u}(\mathbf{t}) + \\ + \mathbf{a_3}(\mathbf{t})\dot{\mathbf{u}}(\gamma(\mathbf{t})) + \mathbf{a_4}(\mathbf{t})\mathbf{u}(\gamma(\mathbf{t})) + \mathbf{f}(\mathbf{t}), \tag{13}$$

$$\mathbf{b_{11}}\dot{\mathbf{u}}(\mathbf{0}) + \mathbf{b_{21}}\mathbf{u}(\mathbf{0}) + \mathbf{c_{11}}\dot{\mathbf{u}}(\mathbf{T}) + \mathbf{c_{21}}\mathbf{u}(\mathbf{T}) = \mathbf{d_1}, \tag{14}$$

$$\mathbf{b_{12}}\dot{\mathbf{u}}(\mathbf{0}) + \mathbf{b_{22}}\mathbf{u}(\mathbf{0}) + \mathbf{c_{12}}\dot{\mathbf{u}}(\mathbf{T}) + \mathbf{c_{22}}\mathbf{u}(\mathbf{T}) = \mathbf{d_2}, \tag{15}$$

where $\mathbf{u}(\mathbf{t})$ is unknown function,

the functions $\mathbf{a_i}(\mathbf{t})$, $\mathbf{i} = \overline{\mathbf{1}, \mathbf{4}}$ and $\mathbf{f}(\mathbf{t})$ are continuous on $[\mathbf{0}, \mathbf{T}]$;

$\gamma(\mathbf{t}) = \zeta_\mathbf{j}$ if $\mathbf{t} \in [\theta_\mathbf{j}, \theta_\mathbf{j+1})$, $\mathbf{j} = \overline{\mathbf{0}, \mathbf{N}-\mathbf{1}}$;

$\theta_\mathbf{j} \le \zeta_\mathbf{j} \le \theta_\mathbf{j+1}$ for all $\mathbf{j} = \mathbf{0}, \mathbf{1}, ..., \mathbf{N}-\mathbf{1}$;

$\mathbf{0} = \theta_\mathbf{0} < \theta_\mathbf{1} < ... < \theta_\mathbf{N-1} < \theta_\mathbf{N} = \mathbf{T}$,

$\mathbf{b_{sp}}$, $\mathbf{c_{sp}}$ and $\mathbf{d_s}$ are constants, where $\mathbf{s}, \mathbf{p} = \mathbf{1}, \mathbf{2}$.

A function $\mathbf{u}(\mathbf{t})$ is a solution to problem (13)-(15) if:

(i) $\mathbf{u}(\mathbf{t})$ is continuously differentiable on $[\mathbf{0}, \mathbf{T}]$;

(ii) the second derivative $\ddot{\mathbf{u}}(\mathbf{t})$ exists at each point $\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$ with the possible exception of the points $\theta_\mathbf{j}$, $\mathbf{j} = \overline{\mathbf{0}, \mathbf{N}-\mathbf{1}}$, where the one-sided derivatives exist;

(iii) equation (13) is satisfied for $\mathbf{u}(\mathbf{t})$ on each interval $(\theta_\mathbf{j}, \theta_\mathbf{j+1})$, $\mathbf{j} = \overline{\mathbf{0}, \mathbf{N}-\mathbf{1}}$, and it holds for the right second derivative of $\mathbf{u}(\mathbf{t})$ at the points $\theta_\mathbf{j}$, $\mathbf{j} = \overline{\mathbf{0}, \mathbf{N}-\mathbf{1}}$;

(iv) boundary conditions (14), (15) are satisfied for $\mathbf{u}(\mathbf{t})$ and $\dot{\mathbf{u}}(\mathbf{t})$ at the points $\mathbf{t} = \mathbf{0}$, $\mathbf{t} = \mathbf{T}$.

We can reduce the two-point boundary value problem for second order DEPCAG (13)–(15) to a problem for system of two DEPCAG.

For this we introduce a new functions $\mathbf{x^{(1)}(t) = u(t)}$, $\mathbf{x^{(2)}(t) = \dot{u}(t)}$, and rewrite of problem (13)–(15) in the form:

$$\mathbf{\dot{x} = A(t)x(t) + A_0(t)x(\gamma(t)) + g(t)}, \tag{16}$$

$$\mathbf{Bx(0) + Cx(T) = d}. \tag{17}$$

where $\mathbf{x(t) = col(x_{(1)}(t), x_{(2)}(t))}$ is unknown vector function,

$$\mathbf{A(t)} = \begin{pmatrix} 0 & 1 \\ a_2(t) & a_1(t) \end{pmatrix}, \qquad \mathbf{A_0(t)} = \begin{pmatrix} 0 & 0 \\ a_4(t) & a_3(t) \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} b_{21} & b_{11} \\ b_{22} & b_{12} \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} c_{21} & c_{11} \\ c_{22} & c_{12} \end{pmatrix},$$

$$\mathbf{g(t)} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \qquad \mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.$$

A vector function $\mathbf{x}(\mathbf{t}) = \mathbf{col}(\mathbf{x}_{(1)}(\mathbf{t}), \mathbf{x}_{(2)}(\mathbf{t}))$ is a solution to problem (16), (17) if:

(i) $\mathbf{x}(\mathbf{t})$ is continuous on $[\mathbf{0}, \mathbf{T}]$;

(ii) the derivative $\dot{\mathbf{x}}(\mathbf{t})$ exists at each point $\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$ with the possible exception of the points $\theta_{\mathbf{j}}, \quad \mathbf{j} = \overline{\mathbf{0}, \mathbf{N} - \mathbf{1}}$, where the one-sided derivatives exist;

(iii) equation (16) is satisfied for $\mathbf{x}(\mathbf{t})$ on each interval $(\theta_{\mathbf{j}}, \theta_{\mathbf{j}+\mathbf{1}})$, $\mathbf{j} = \overline{\mathbf{0}, \mathbf{N} - \mathbf{1}}$, and it holds for the right derivative of $\mathbf{x}(\mathbf{t})$ at the points $\theta_{\mathbf{j}}, \quad \mathbf{j} = \overline{\mathbf{0}, \mathbf{N} - \mathbf{1}}$;

(iv) boundary condition (17) is satisfied for $\mathbf{x}(\mathbf{t})$ at the points $\mathbf{t} = \mathbf{0}$, $\mathbf{t} = \mathbf{T}$.

For problem (16), (17) we can apply the results in Sections 5 and 6.

Now, we establish solvability conditions of problem (13)–(15) in the terms of initial data. We apply parametrization method to problem (13)–(15).

Let $\mathbf{u_r(t)}$ denote the restriction of the function $\mathbf{u(t)}$ to the $\mathbf{r}$th subinterval $[\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}})$, i.e.

$$\mathbf{u_r(t) = u(t)} \quad \text{for} \quad \mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}}), \quad \mathbf{r} = \overline{\mathbf{1, N}}.$$

We also have that $\dot{\mathbf{u}}_{\mathbf{r}}(\mathbf{t})$ is the restriction of the function $\dot{\mathbf{u}}(\mathbf{t})$ to the $\mathbf{r}$th subinterval $[\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}})$, and

$$\dot{\mathbf{u}}_{\mathbf{r}}(\mathbf{t}) = \dot{\mathbf{u}}(\mathbf{t}) \quad \text{for} \quad \mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}}), \quad \mathbf{r} = \overline{\mathbf{1, N}}.$$

Then the system $\mathbf{u[t] = (u_1(t), u_2(t), \ldots, u_N(t))}$ belongs to the space $\mathbf{C([0, T], \Delta_N, \mathbb{R}^N)}$, and its elements $\mathbf{u_r(t)}$, $\mathbf{r} = \overline{\mathbf{1, N}}$, satisfy the following second order DEPCAG

$$\ddot{\mathbf{u}}_{\mathbf{r}} = \mathbf{a_1(t)}\dot{\mathbf{u}}_{\mathbf{r}}(\mathbf{t}) + \mathbf{a_2(t)u_r(t)} + \mathbf{a_3(t)}\dot{\mathbf{u}}_{\mathbf{r}}(\zeta_{\mathbf{r-1}}) +$$

$$+\mathbf{a_4(t)u_r}(\zeta_{\mathbf{r-1}}) + \mathbf{f(t)}, \qquad \mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}}), \qquad \mathbf{r} = \overline{\mathbf{1, N}}. \quad (18)$$

Conditions (14) and (15) have the form

$$\mathbf{b_{11}\dot{u}_1(0)+b_{21}u_1(0)+c_{11}\lim_{t\to T-0}\dot{u}_N(t)+c_{21}\lim_{t\to T-0}u_N(t)=d_1}, \qquad (19)$$

$$\mathbf{b_{12}\dot{u}_1(0)+b_{22}u_1(0)+c_{12}\lim_{t\to T-0}\dot{u}_N(t)+c_{22}\lim_{t\to T-0}u_N(t)=d_2}, \qquad (20)$$

And, we have continuity conditions of functions $\mathbf{u_p(t)}$ and $\mathbf{\dot{u}_p(t)}$ at the interior points $\mathbf{t}=\theta_\mathbf{p}$, $\mathbf{p}=\mathbf{1,2,...,N-1}$:

$$\lim_{\mathbf{t}\to\theta_\mathbf{p}-\mathbf{0}}\mathbf{u_p(t)}=\mathbf{u_{p+1}}(\theta_\mathbf{p}), \qquad \mathbf{p}=\overline{\mathbf{1,N-1}}, \qquad (21)$$

$$\lim_{\mathbf{t}\to\theta_\mathbf{p}-\mathbf{0}}\mathbf{\dot{u}_p(t)}=\mathbf{\dot{u}_{p+1}}(\theta_\mathbf{p}), \qquad \mathbf{p}=\overline{\mathbf{1,N-1}}. \qquad (22)$$

We introduce parameters

$$\lambda_{\mathbf{r}} = \mathbf{u_r}(\zeta_{\mathbf{r-1}}), \quad \mu_{\mathbf{r}} = \dot{\mathbf{u}}_{\mathbf{r}}(\zeta_{\mathbf{r-1}}), \quad \mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}.$$

By making the substitution

$$\mathbf{z_r}(\mathbf{t}) = \mathbf{u_r}(\mathbf{t}) - \lambda_{\mathbf{r}} - (\mathbf{t} - \zeta_{\mathbf{r-1}})\mu_{\mathbf{r}}, \qquad \dot{\mathbf{z}}_{\mathbf{r}}(\mathbf{t}) = \dot{\mathbf{u}}_{\mathbf{r}}(\mathbf{t}) - \mu_{\mathbf{r}}$$

on each subinterval $[\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}})$, $\quad \mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}$,
we get the second order ODEs with parameters

$$\ddot{\mathbf{z}}_{\mathbf{r}} = \mathbf{a_1}(\mathbf{t})\dot{\mathbf{z}}_{\mathbf{r}}(\mathbf{t}) + \mathbf{a_2}(\mathbf{t})\mathbf{z_r}(\mathbf{t}) + \mathbf{a_1}(\mathbf{t})\mu_{\mathbf{r}} + \mathbf{a_2}(\mathbf{t})\lambda_{\mathbf{r}} +$$

$$+\mathbf{a_2}(\mathbf{t})(\mathbf{t} - \zeta_{\mathbf{r-1}})\mu_{\mathbf{r}} + \mathbf{a_3}(\mathbf{t})\mu_{\mathbf{r}} + \mathbf{a_4}(\mathbf{t})\lambda_{\mathbf{r}} + \mathbf{f}(\mathbf{t}), \qquad (23)$$

$$\mathbf{t} \in [\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}}), \qquad \mathbf{r} = \overline{\mathbf{1}, \mathbf{N}},$$

subject to the initial conditions

$$\mathbf{z_r}(\zeta_{\mathbf{r-1}}) = \mathbf{0}, \qquad \dot{\mathbf{z}}_{\mathbf{r}}(\zeta_{\mathbf{r-1}}) = \mathbf{0}, \qquad \mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}. \qquad (24)$$

Thus we obtain the Cauchy problems (23), (24) for second order ODEs with parameters on the subintervals $[\theta_{\mathbf{r-1}}, \theta_{\mathbf{r}})$, $\mathbf{r} = \overline{\mathbf{1}, \mathbf{N}}$.

Conditions (19)–(22) have following form

$$[\mathbf{b_{11}}+\mathbf{b_{21}}(\mathbf{0}-\zeta_\mathbf{0})]\mu_\mathbf{1}+\mathbf{b_{21}}\lambda_\mathbf{1}+[\mathbf{c_{11}}+\mathbf{c_{21}}(\mathbf{T}-\zeta_\mathbf{N-1})]\mu_\mathbf{N}+\mathbf{c_{21}}\lambda_\mathbf{N} =$$

$$= \mathbf{d_1}-\mathbf{b_{11}}\dot{\mathbf{z}}_\mathbf{1}(\mathbf{0})-\mathbf{b_{21}}\mathbf{z_1}(\mathbf{0})-\mathbf{c_{11}}\lim_{\mathbf{t}\to\mathbf{T}-\mathbf{0}}\dot{\mathbf{z}}_\mathbf{N}(\mathbf{t})-\mathbf{c_{21}}\lim_{\mathbf{t}\to\mathbf{T}-\mathbf{0}}\mathbf{z_N}(\mathbf{t}), \tag{25}$$

$$[\mathbf{b_{12}}+\mathbf{b_{22}}(\mathbf{0}-\zeta_\mathbf{0})]\mu_\mathbf{1}+\mathbf{b_{22}}\lambda_\mathbf{1}+[\mathbf{c_{12}}+\mathbf{c_{22}}(\mathbf{T}-\zeta_\mathbf{N-1})]\mu_\mathbf{N}+\mathbf{c_{22}}\lambda_\mathbf{N} =$$

$$= \mathbf{d_2}-\mathbf{b_{12}}\dot{\mathbf{z}}_\mathbf{1}(\mathbf{0})-\mathbf{b_{22}}\mathbf{z_1}(\mathbf{0})-\mathbf{c_{12}}\lim_{\mathbf{t}\to\mathbf{T}-\mathbf{0}}\dot{\mathbf{z}}_\mathbf{N}(\mathbf{t})-\mathbf{c_{22}}\lim_{\mathbf{t}\to\mathbf{T}-\mathbf{0}}\mathbf{z_N}(\mathbf{t}), \tag{26}$$

$$\lambda_\mathbf{p} + (\theta_\mathbf{p} - \zeta_\mathbf{p-1})\mu_\mathbf{p} - \lambda_\mathbf{p+1} + (\theta_\mathbf{p} - \zeta_\mathbf{p})\mu_\mathbf{p+1} =$$

$$= -\lim_{\mathbf{t}\to\theta_\mathbf{p}-\mathbf{0}}\mathbf{z_p}(\mathbf{t}) + \mathbf{z_{p+1}}(\theta_\mathbf{p}), \qquad \mathbf{p} = \overline{\mathbf{1}, \mathbf{N}-\mathbf{1}}, \tag{27}$$

$$\mu_\mathbf{p} - \mu_\mathbf{p+1} = -\lim_{\mathbf{t}\to\theta_\mathbf{p}-\mathbf{0}}\dot{\mathbf{z}}_\mathbf{p}(\mathbf{t}) + \dot{\mathbf{z}}_\mathbf{p+1}(\theta_\mathbf{p}), \qquad \mathbf{p} = \overline{\mathbf{1}, \mathbf{N}-\mathbf{1}}. \tag{28}$$

Let $\alpha_r(t) = \int\limits_{\zeta_{r-1}}^{t} a_1(\tau)d\tau$, $r = \overline{1, N}$.

For $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$, the Cauchy problem (23), (24) is equivalent to Volterra integral equation second kind :

$$z_r(t) = \int\limits_{\zeta_{r-1}}^{t} e^{\alpha_r(\tau)} \int\limits_{\zeta_{r-1}}^{\tau} e^{-\alpha_r(s)} a_2(s) z_r(s)\Big] ds d\tau +$$

$$+ \int\limits_{\zeta_{r-1}}^{t} e^{\alpha_r(\tau)} \int\limits_{\zeta_{r-1}}^{\tau} e^{-\alpha_r(s)} \Big[a_2(s) + a_4(s)\Big] ds d\tau \lambda_r +$$

$$+ \int\limits_{\zeta_{r-1}}^{t} e^{\alpha_r(\tau)} \int\limits_{\zeta_{r-1}}^{\tau} e^{-\alpha_r(s)} \Big[a_1(s) + a_2(s)(s - \zeta_{r-1}) + a_3(s)\Big] ds d\tau \mu_r +$$

$$+ \int\limits_{\zeta_{r-1}}^{t} e^{\alpha_r(\tau)} \int\limits_{\zeta_{r-1}}^{\tau} e^{-\alpha_r(s)} f(s) ds d\tau. \qquad (29)$$

From (29), we find $\mathbf{z_1(0)}$, $\mathbf{\dot{z}_1(0)}$, $\lim\limits_{\mathbf{t\to T-0}} \mathbf{z_N(t)}$, $\lim\limits_{\mathbf{t\to T-0}} \mathbf{\dot{z}_N(t)}$, $\lim\limits_{\mathbf{t\to\theta_p-0}} \mathbf{z_p(t)}$, $\mathbf{z_{p+1}(\theta_p)}$, $\lim\limits_{\mathbf{t\to\theta_p-0}} \mathbf{\dot{z}_p(t)}$, $\mathbf{\dot{z}_{p+1}(\theta_p)}$, $\mathbf{p=\overline{1,N-1}}$. Substituting them into (25)–(28), we get the system of algebraic equations in unknown parameters $\lambda_\mathbf{r}$, and $\mu_\mathbf{r}$, $\mathbf{r=\overline{1,N}}$:

$$\mathbf{Q(\Delta_N)}\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = -\mathbf{F(\Delta_N)} - \mathbf{G(\Delta_N, z)}. \qquad (30)$$

The matrix $Q(\Delta_N)$ composed by coefficients of parameters and has a special block-banded structure, maps $\mathbb{R}^{2N}$ into $\mathbb{R}^{2N}$.

We construct **Algorithm B** for finding solution to problem (23)–(28). The algorithm consists of two parts:
(1) the values of the unknown parameters $\lambda_\mathbf{r}$ and $\mu_\mathbf{r}$ are found from the system of algebraic equations (30);
(2) the unknown function $\mathbf{z_r(t)}$ are found from the Cauchy problems (23), (24) for second order ODEs.

Introduce notation

$$\alpha_1 = \max\Big(1, \max_{t\in[0,T]} |a_1(t)| + \max_{t\in[0,T]} |a_2(t)|\Big),$$

$$\alpha_2 = \max_{t\in[0,T]} |a_3(t)| + \max_{t\in[0,T]} |a_4(t)|,$$

$$\theta = \max\Big\{ \max_{r=\overline{1,N}} (\theta_r - \zeta_{r-1}), \max_{r=\overline{1,N}} (\zeta_{r-1} - \theta_{r-1}) \Big\}.$$

$$\beta = \max\Big(|b_{21}| + |b_{11}|, |b_{22}| + |b_{12}|\Big),$$

$$\gamma = \max\Big(|c_{21}| + |c_{11}|, |c_{22}| + |c_{12}|\Big).$$

The following statement is true.

THEOREM 5.

Suppose that, the matrix $Q(\Delta_N) : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is invertible and the following inequalities hold:

(a) $||[Q(\Delta_N)]^{-1}|| \le \chi(\Delta_N)$, *where* $\chi(\Delta_N)$ *is a positive constant;*

(b) $q(\Delta_N) = \chi(\Delta_N) \max(\beta + \gamma, 2)\Big[e^{\alpha_1\theta} - 1 - \alpha_1\theta\Big] < 1.$

Then problem (13)–(15) has a unique solution.

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Thank you for your attention!