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Nash Equilibria in certain two-choice multi-player games played on the ladder graph *

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GRAPHICAL GAMES*

*Different than *graph games*!

- M. Kearns *et al.* [1] (2001)
- Graph and Game
  - Vertices/nodes ~ players
  - Edges/lines ~ connections btw players to play the game

Find the equilibrium configuration/s and analyse them

[1] **Graphical Models for Game Theory**, *Proceedings of 17th Conference of Uncertainty in Artificial Intelligence*

GRAPHICAL GAMES

- Applications
  - (Evolutionary) Biology → Allen *et al.* [2] (2019)
  - Economics → Leduc *et al.* [3] (2017)
  - Sociology → Eger *et al.* [4] (2016)
  - Computer Science → Cibulka *et al.* [5] (2013)

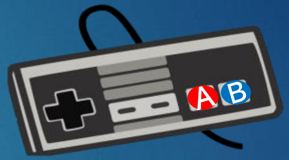
[2] **Evolutionary games on isothermal graphs**, Nature Communications
[3] **Strategic investment in protection in networked systems**, *Network Science*
[4] **Opinion dynamics and wisdom under out-group discrimination**, *Mathematical Social Sciences*
[5] **Graph sharing games: Complexity and connectivity**, *Theoretical Computer Science*

OUR GAME

PLAYER 1

PLAYER 2

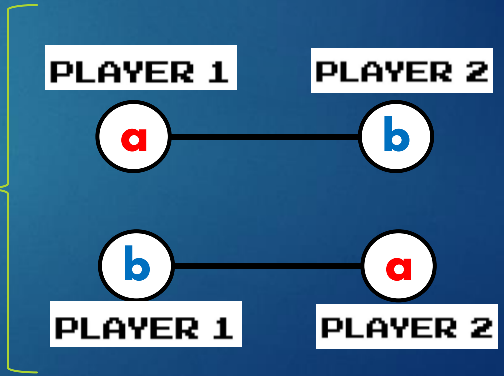
2 players
2 choices



		PLAYER 2	
		A	B
PLAYER 1	A	(p,p)	(q,r)
	B	(r,q)	(s,s)

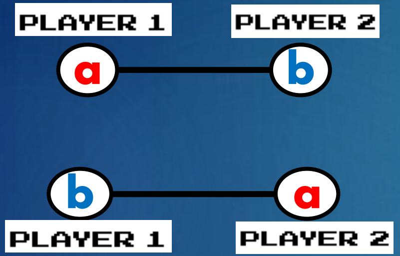
$r > p$
$q > s$

2 Nash Equilibrium (NE) solutions

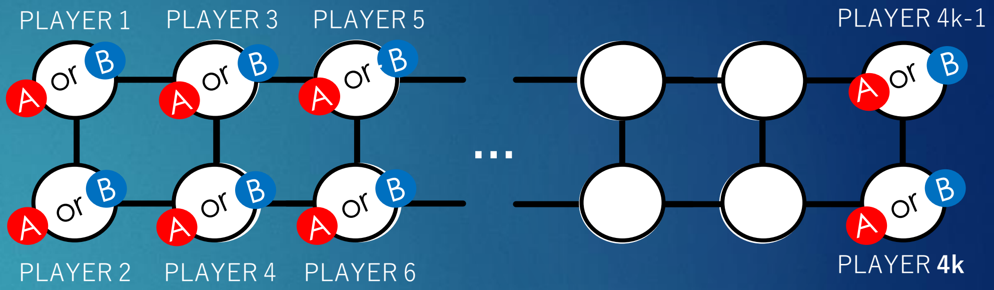


(anti-coordination game)

OUR GRAPH/S

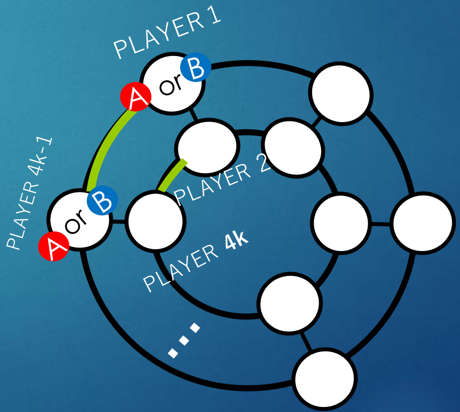


Ladder graph



For simplicity 4k players

Circular ladder



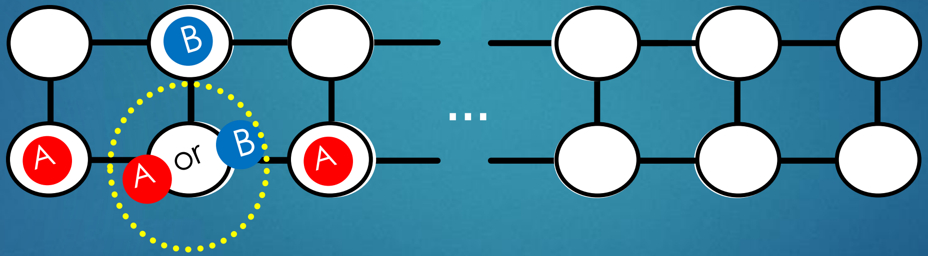
How many NE do we have now?

NE in ladder

How many NE do we have now?

What's best response for Player i ?

Ladder graph



- Average payoff when playing A
- Average payoff when playing B

depend on payoff parameters (r, p, q, s)

Which payoff is higher? → best response

NE in ladder

Given opponent's strategy s_{-i}

- Average payoff when playing A
- Average payoff when playing B

depend on payoff parameters

Which payoff is higher? → best response

Further assumptions on payoff parameters in addition to $r > p$, $q > s$

Ladder best response Player i

4 cases to define best reponse

$x \equiv r - p > 0$

$y \equiv q - s > 0$

2 different cases

$x/2 > y$	
s_{-i}	s_i^*
aaa	b
bbb	a
aab	b
abb	b
aa	b
bb	a
ab	b

$x > y > x/2$	
s_{-i}	s_i^*
aaa	b
bbb	a
aab	b
abb	a
aa	b
bb	a
ab	b

$2x > y > x$	
s_{-i}	s_i^*
aaa	b
bbb	a
aab	b
abb	a
aa	b
bb	a
ab	a

$y > 2x$	
s_{-i}	s_i^*
aaa	b
bbb	a
aab	a
abb	a
aa	b
bb	a
ab	a

Interchange a ↔ b

r, p, q, s → *payoff parameters*

CASE 1: NE in ladder

$x \equiv r - p > 0$

$y \equiv q - s > 0$

$r, p, q, s \rightarrow$ payoff parameters

Best response case 1

$x > y > x/2$	
s_{-i}	s_i^*
aaa	b
bbb	a
aab	b
abb	a
aa	b
bb	a
ab	b

start to build
some solutions

Block 0

a b

b a

Block 1

b a

a b

Block 2

a a

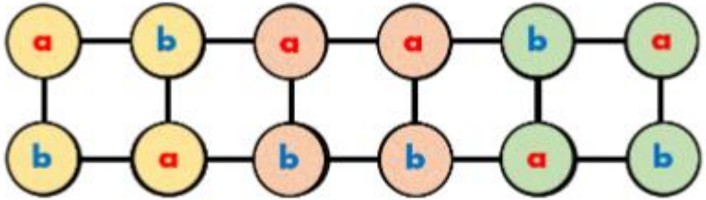
b b

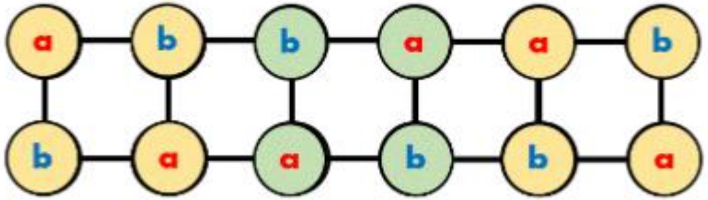
Block 3

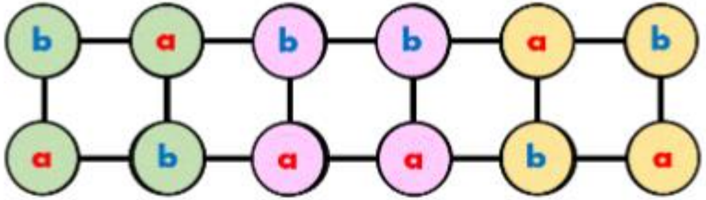
b b

a a

three NE solutions for ladder with 4k=12 players $(k = 3)$

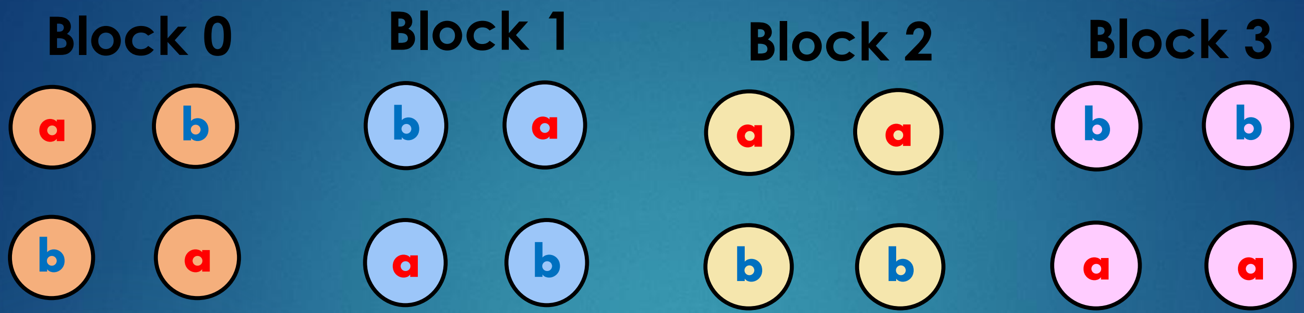






$4k \rightarrow$ no. players
$k \rightarrow$ no. blocks used

CASE 1: NE in ladder



$x \equiv r - p > 0$
$y \equiv q - s > 0$

$r, p, q, s \rightarrow$ *payoff parameters*

Best response case 1

$x > y > x/2$	
s_{-i}	s_i^*
aaa	b
bbb	a
aab	b
abb	a
aa	b
bb	a
ab	b

rules to attach the blocks

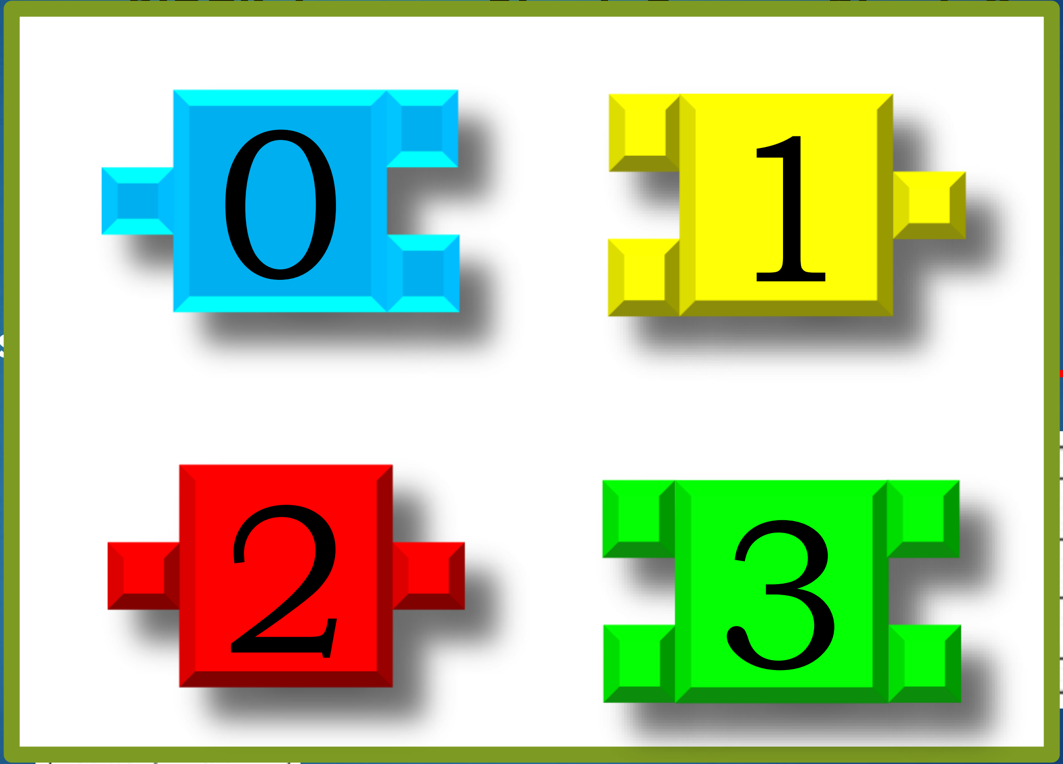
0, 1, 3	$\rightarrow$	**0**	$\rightarrow$	0, 1, 2
0, 1, 2	$\rightarrow$	**1**	$\rightarrow$	0, 1, 3
0, 3	$\rightarrow$	**2**	$\rightarrow$	3, 1
1, 2	$\rightarrow$	**3**	$\rightarrow$	0, 2

CASE 1: NE in ladder

Block 0

a b
b a

Bes



the blocks

→	0, 1, 2
→	0, 1, 3
→	3, 1
→	0, 2

$$x \equiv r - p > 0$$
$$y \equiv q - s > 0$$

$r, p, q, s \rightarrow$ *payoff parameters*

ab	b

CASE 1: NE in ladder

Example of one solution for case 1 using $k = 7$ blocks (28 players)

CASE 1: NE in ladder (recap)

2-choice 2-player game with 2 NE

+ payoff parameters

Ladder graph

How many NE do we have now?

Define best response

Depending on inequalities btw payoff parameters

2 different cases when we want to count no. NE

CASE 1

All NE solutions are built out of 4 elementary blocks

+ some rules to attach them

CASE 1: NE in ladder (recap)

2-choice 2-player game with 2 NE

+ payoff parameters

Ladder graph

How many NE do we have now?

Define best response

Depending on inequalities btw payoff parameters

2 different cases when we want to count no. NE

CASE 1

All … uilt
o … y

+ s … ch
them

How many NE do we have now?

count how many combinations of blocks given the rules are possible

CASE 1: NE in ladder

For simplicity, assume no. players is a multiple of 4 $(\mathrm{i.e.}\ 4k)$

$4k \rightarrow$ no. players
$k \rightarrow$ no. blocks used

$$N_{..j}(k) \qquad j \in \{0,1,2,3\}$$

Total no. NE

$$N(k) = N_{..0}(k) + N_{..1}(k) + N_{..2}(k) + N_{..3}(k)$$

using the rules to attach the blocks

$$N_{..0}(k) = N_{..0}(k-1) + N_{..1}(k-1) + N_{..3}(k-1)$$
$$N_{..1}(k) = N_{..0}(k-1) + N_{..1}(k-1) + N_{..2}(k-1)$$
$$N_{..2}(k) = N_{..0}(k-1) + N_{..3}(k-1)$$
$$N_{..3}(k) = N_{..1}(k-1) + N_{..2}(k-1)$$

Example of one solution for case 1 using $k = 7$ blocks (28 players)

$$N(k) = 3N(k-1) - N(k-2)$$

CASE 1: NE in ladder

$$N(k) = 3N(k-1) - N(k-2)$$

$N(k) \sim cr^k$

/…/

$$N(k) = \alpha\varphi^{2k} + \beta\varphi^{-2k}$$

golden ratio

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1,62$$

$\alpha, \beta \rightarrow$ initial conditions

Initial conditions

/…/

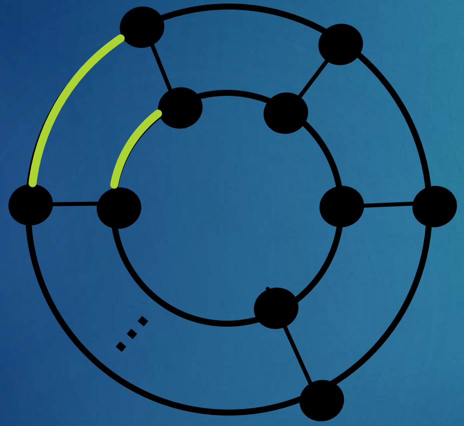
Total no. NE

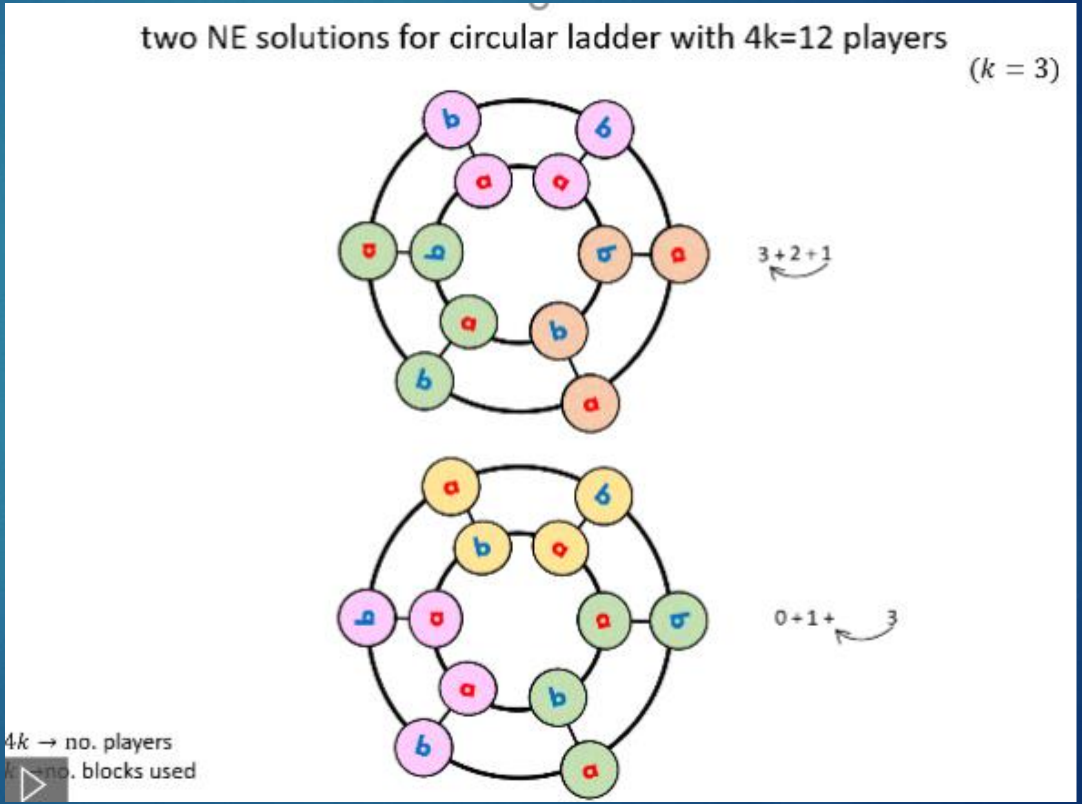
$$N_{ladder}(4k) = \frac{2}{\sqrt{5}}\left[\varphi^{2k-1} + \varphi^{-(2k-1)}\right]$$

$4k \rightarrow$ no. players
$k \rightarrow$ no. blocks used

CASE 1: NE in circular ladder

Circular ladder





CASE 1: NE in circular ladder

rules to attach the blocks

Example of one solution for case 1 using $k = 7$ blocks (28 players)

$0 + \cdots + 0$ $\quad$ $0 + \cdots + 1$ $\quad$ $0 + \cdots + 2$ $\quad$ $0 + \cdots + 3$

From rules, we cannot attach 2+0

We need to count the number of allowed combinations when matching endings:

- $0 + \cdots + 0$; $0 + \cdots + 1$; $0 + \cdots + 3$;
- $1 + \cdots + 0$; $1 + \cdots + 1$; $1 + \cdots + 2$;
- $2 + \cdots + 0$; $2 + \cdots + 3$
- $3 + \cdots + 1$; $3 + \cdots + 2$

Total no. NE

$$N_{circ}(k) = N_{0..0}(k) + N_{0..1}(k) + N_{0..3}(k) + N_{1..0}(k) + N_{1..1}(k) + N_{1..2}(k) + N_{2..0}(k) + N_{2..3}(k) + N_{3..1}(k) + N_{3..2}(k)$$

CASE 1: NE in circular ladder

$$N_{circ}(k) = N_{0..0}(k) + N_{0..1}(k) + N_{0..3}(k)$$
$$+N_{1..0}(k) + N_{1..1}(k) + N_{1..2}(k)$$
$$+ N_{2..0}(k) + N_{2..3}(k)$$
$$+N_{3..1}(k) + N_{3..2}(k)$$

$4k \rightarrow$ no. players
$k \rightarrow$ no. blocks used

$N_{i..j}(k)$ → $N_{..j}(k)$ Building solutions attaching blocks **to the right** of last block

$i,j \in \{0,1,2,3\}$ → $N_{i..}(k)$ Building solutions attaching blocks **to the left** of last block

...some technical details → $N_{i..j}(k)$ **+ initial conditions/solutions**

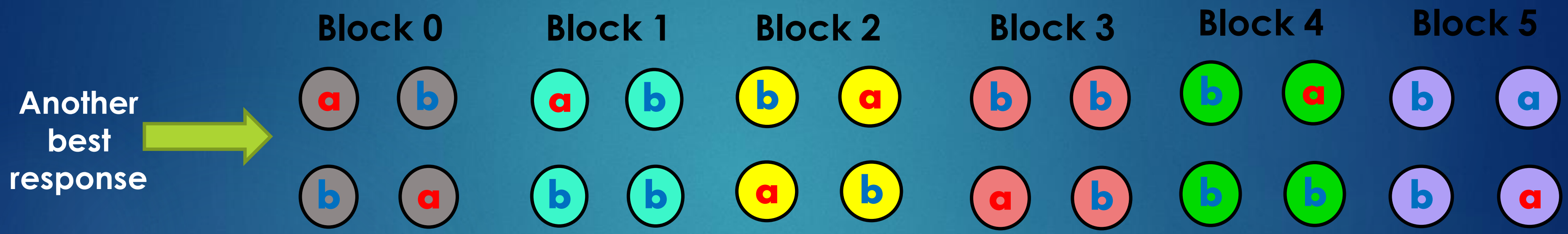
/.../

$$N_{circular}(4k) = \varphi^{2k-1} + \varphi^{-(2k-1)} + 2$$

golden ratio

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1,62$$

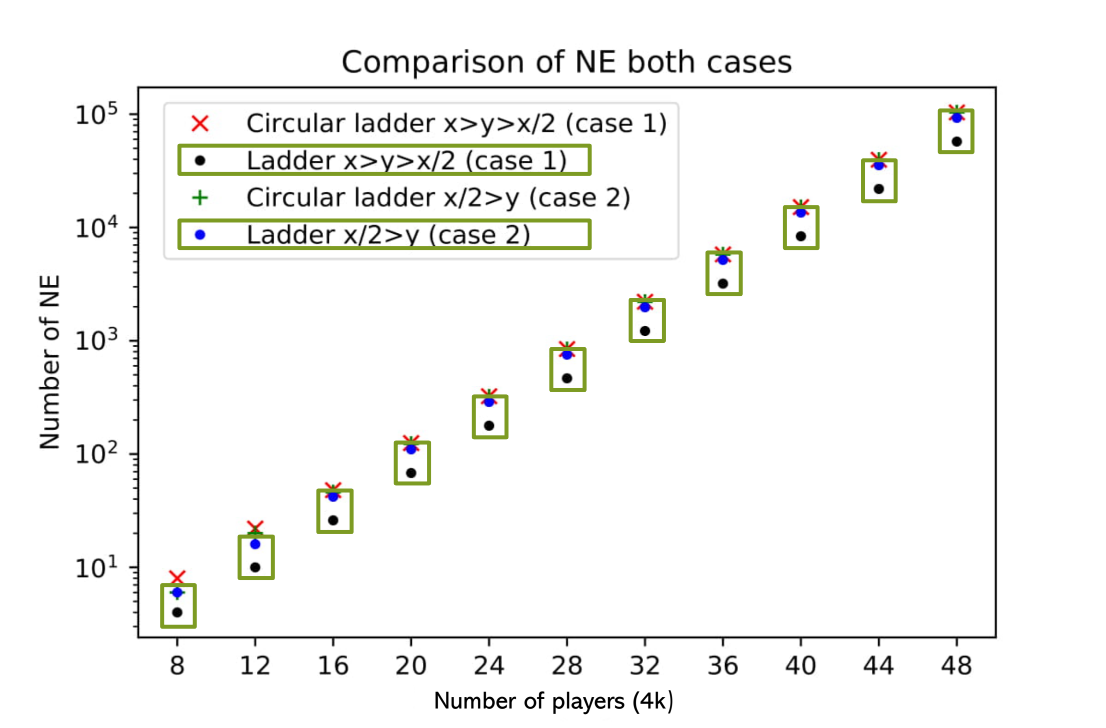
CASE 2: NE in ladder and circular ladder



+ other rules to attach the blocks

Same procedure as before → find same recurrence relation…

Comparison of results



CASE 1

$$N_{ladder}(4k) = \frac{2}{\sqrt{5}}\left[\varphi^{2k-1} + \varphi^{-(2k-1)}\right]$$

$$N_{circular}(4k) = \varphi^{2k} + \varphi^{-2k} + 2$$

CASE 2

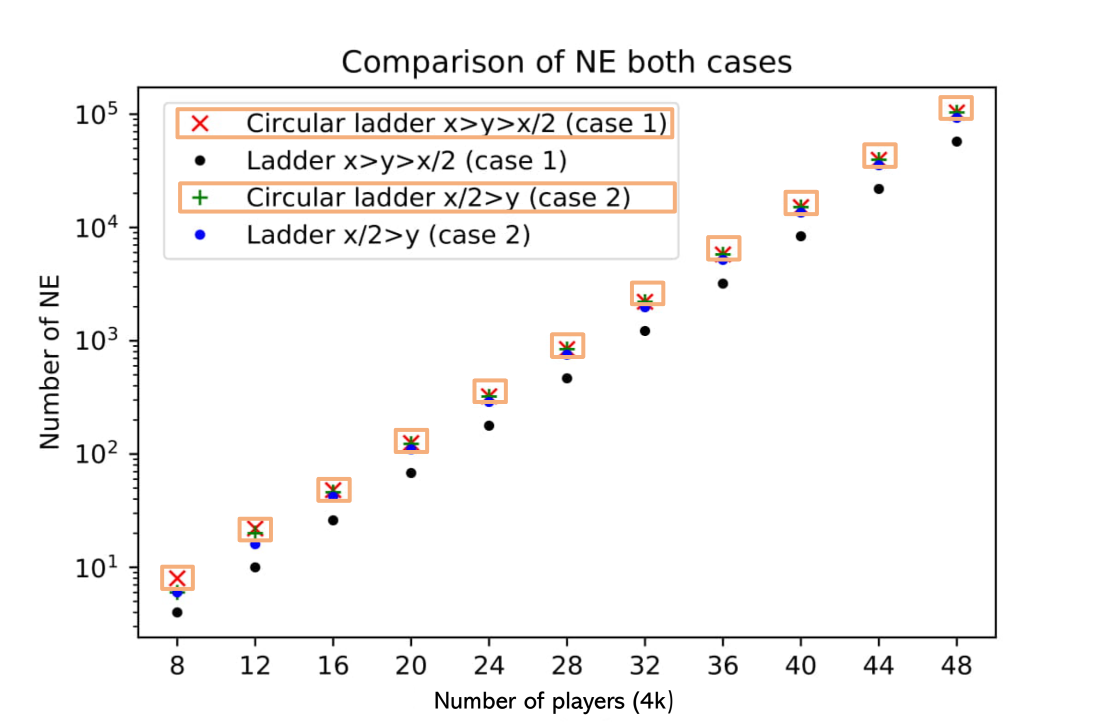
$$N_{ladder}(4k) = \frac{2}{\sqrt{5}}\left[\varphi^{2k} + \varphi^{-(2k)}\right]$$

$$N_{circular}(4k) = \varphi^{2k} + \varphi^{-2k}$$

golden ratio

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1{,}62\ldots$$

Comparison of results



CASE 1

$$N_{ladder}(4k) = \frac{2}{\sqrt{5}}\left[\varphi^{2k-1} + \varphi^{-(2k-1)}\right]$$

$$N_{circular}(4k) = \varphi^{2k} + \varphi^{-2k} + 2$$

CASE 2

$$N_{ladder}(4k) = \frac{2}{\sqrt{5}}\left[\varphi^{2k} + \varphi^{-(2k)}\right]$$

$$N_{circular}(4k) = \varphi^{2k} + \varphi^{-2k}$$

golden ratio

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1{,}62\ldots$$

Conclusions

For 2-choice 2-player anti-coordination game and (circular) ladder

- Found explicit analytic formulae for NE
- Golden ratio base of exponential growth
- NE circular > NE ladder
- NE ladder changes with case (relation btw payoff parameters)
- NE circular ≈ const regardless of case → topology of graph plays a role?

Further work/open questions

- Are the NE configurations Pareto Optimal?
- What if coordination game, same results?
- Same game but another regular graph
- …

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