CHAIN CONNECTED PAIR OF A TOPOLOGICAL SPACE AND ITS SUBSPACE

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1. INTRODUCTION

1. The definition of connectedness, that is considered as standard, was first given in the beginning of 20th century by Riesz and Hausdorff: The subset Y of a topological space X is connected, if Y cannot be expressed as a union of two nonempty separated subsets (The nonempty sets $A, B \subseteq X$ are separated if $\overline{A} \cap B = A \cap \overline{B} = \varnothing$). A disadvantage of this definition is that it is given with a negative sentence.

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2. In 1883, Cantor gave a definition of connectedness in $\mathbb{R}^n$ by using the notion of chain. Later the definition is generalized to all topological spaces: The topological space X is connected if for every $x, y \in X$ and for any covering $\mathcal{U}$ of X, there exists a chain in $\mathcal{U}$ that connects x and y. Where by a covering we understand an open covering and a chain in $\mathcal{U}$ that connects x and y (from x to y (y to x)) is a finite sequence $U_1, U_2, \ldots, U_n$ in $\mathcal{U}$ such that $U_i \cap U_{i+1} \neq \varnothing$ for every $i = 1, 2, \ldots, n-1$, $x \in U_1$ and $y \in U_n$.

2 CHAIN CONNECTED SETS

Definition 2.1. Let X be a topological space and $C \subseteq X$. The set C is **chain connected** in X, if for every covering $\mathcal{U}$ of X in X and every $x, y \in C$, there exists a chain in $\mathcal{U}$ that connects x and y.

Let X be a topological space and $C \subseteq Y \subseteq X$.

Theorem 2.2. If C is chain connected in Y (C), then C is chain connected in X.

Example 2.3. Let

$$X = \left\{ \left(x, \frac{1}{n} \right) \middle| x \in [-1,1], n \in \mathbb{N} \right\} \cup \left\{ (x,0) \middle| x \in [-1,0) \cup (0,1] \right\}$$

be a topological space.

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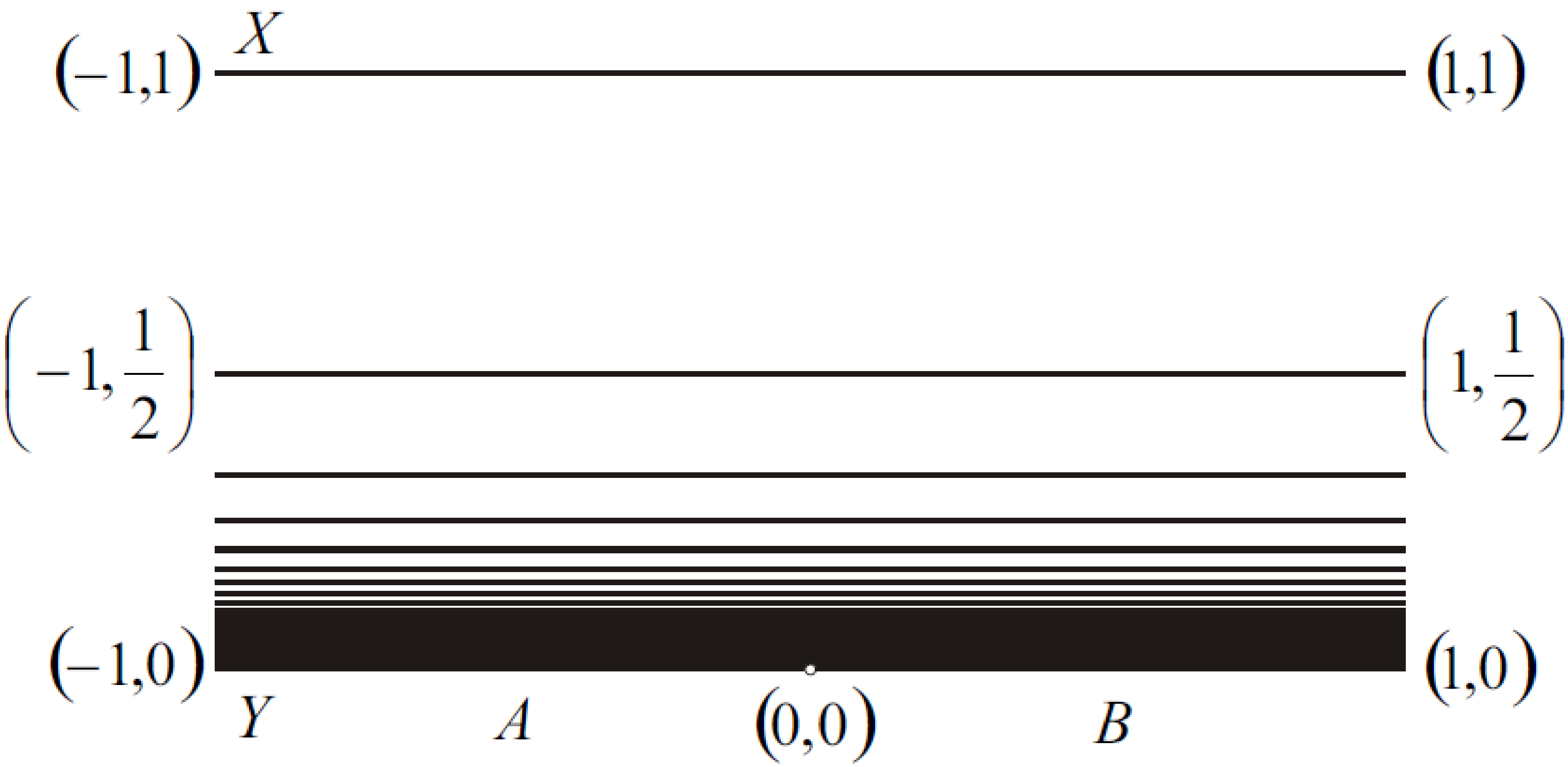
Let $A=[-1,0)\times\{0\}$ and $B=(0,1]\times\{0\}$. Then $Y=A\cup B$ is chain connected in X, but is not chain connected in Y.



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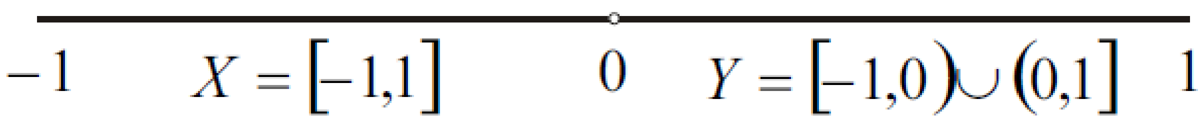
The converse claim does not hold for a simpler space either, for example an interval.

Example 2.4. The space $X=[-1,1]$ is compact and metric. Let $A=[-1,0)$, $B=(0,1]$ and $Y=A\cup B$. Then Y is chain connected in X, but it is not chain connected in Y. Moreover Y is not connected.



Remark 2.5. If the set C is chain connected in X, then each subset of C is chain connected in X.■

Let X be a topological space, $\mathcal{U}$ be a covering of X in X, and $x\in C\subseteq X$.

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We denote by $A_{CX}(x,\mathcal{U})$ the set that consists of all elements $y \in C$ such that there exists a chain in $\mathcal{U}$, that connects x and y. If $C = X$, $A_X(x,\mathcal{U}) = A_{XX}(x,\mathcal{U})$.

Example 2.6. For spaces A, B, Y, X from example 2.2, $A_X(x,\mathcal{U}) = X$ and $A_{YX}(x,\mathcal{U}) = Y$.

Theorem 2.7. The set $A_{CX}(x,\mathcal{U})$ is nonempty, open, and closed in C.

The following theorem gives the relationship between a connected and a chain connected set.

Theorem 2.8. A set C is connected if and only if C is chain connected in C.

3. CHAIN SEPARATED SETS

Chain separated sets will be defined in such a way that the notions of chain connectedness and chain separatedness are analogous to the notions of connectedness and separatedness.

Let X be a topological space, and let A and B be nonempty subsets of X.

Definition 3.1. The sets A and B are **chain separated in** X, if there exists a covering $\mathcal{U}$ of X in X such that for every point $x \in A$ and every $y \in B$, there is no chain in $\mathcal{U}$ that connects x and y.

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From the definition, it follows that if A and B are chain separated in a topological space X, then any two sets C and D, where $C \subseteq A$ and $D \subseteq B$, are chain separated in X.

Let X be a topological space, let $Y \subseteq X$, and let A and B be nonempty subsets of X.

Proposition 3.2. If A and B are chain separated in X, then A and B are chain separated in Y ($A \cup B$).■

Example 3.3. If A, B, Y, X are topological spaces from example 2.2, then A and B are chain separated in $Y = A \cup B$, but they are not chain separated in X.

Theorem 3.4. Nonempty sets A and B are chain separated in $A \cup B$, if and only if A and B are separated.

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Now let us consider two statements that give criteria for connected and chain connected sets by using chain separatedness. Let X be a topological space and $C \subseteq X$.

Theorem 3.5. A set C is chain connected in X, if and only if C cannot be represented as a union of two chain separated sets A and B in X.

Corollary 3.6. A set C is connected, if and only if it cannot be represented as a union of two chain separated sets A and B in C.■

Theorem 3.7. Let $X = A \cup B$, where A and B are chain separated sets in X, and C is a chain connected set in X. Then $C \subseteq A$ or $C \subseteq B$.

Corollary 3.8. Let $X = A \cup B$, where A and B are separated. If C is a connected set, then $C \subset A$ or $C \subset B$.■

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Theorem 3.9. If C is chain connected in X and $f:X\to Y$ is a continuous function, then $f(C)$ is chain connected in $f(X)$.

Remark 3.10. If C is a connected set and $f:X\to Y$ is a continuous function, then $f(C)$ is connected.■

Theorem 3.11. Let $C\subseteq D\subseteq \overline{C}\subseteq X$. Set C is chain connected in X, if and only if D ($\overline{C}$) is chain connected in X.

Corollary 3.12. Let X be a topological space and $C\subseteq X$. If C is a connected set and $C\subseteq D\subseteq \overline{C}$, then D is connected.

The next example shows that the reverse claim does not have to be valid.

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Example 3.13. Let $X = [-1,1]$. Set $C = [-1,0) \cup (0,1]$ is not connected, but $\overline{C} = [-1,1]$ is connected.

Lemma 3.14. Let $C, D \subseteq X$. If C and D are chain connected in X and $\overline{C} \cap \overline{D} \neq \emptyset$, then the union $\overline{C} \cup \overline{D}$ is chain connected in X.

Theorem 3.15. Let $C_i, i \in I$ be a family of chain connected subspaces of X. If there exists $i_0 \in I$ such that for every $i \in I$, $\overline{C}_{i_0} \cap \overline{C}_i \neq \emptyset$, then the union $\bigcup_{i \in I} \overline{C}_i$ is chain connected in X.

Corollary 3.16. Let $C_i \subseteq X, i \in I$ be a family of connected sets. If there exists $i_0 \in I$ such that for every $i \in I$,

$C_{i_0} \cap C_i \neq \emptyset$, then the union $\bigcup_{i \in I} C_i$ is connected.■

It is clear that <u>if</u> every two points x and y of X are in a chain connected set C_{xy} in X, then X is chain connected. Therefore:

Corollary 3.17. If every two points x and y of X are in a connected set C_{xy} in X, then X is connected.■

Theorem 3.18. If A and B are chain separated sets in X, then there exist:

a) open neighbourhoods U and V that separate them, i.e. $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$

b) closed neighbourhoods F and G that separate them, i.e. $A \subseteq F$, $B \subseteq G$, and $F \cap G = \emptyset$.

4. CRITERIA FOR CHAIN CONNECTEDNESS AND CONNECTEDNESS USING NOTIONS OF CONTINUOUS FUNCTION OR STAR OF A COVERING. CHAIN CONNECTEDNESS RELATION

Let X be a topological space, $x \in X$ and $\mathcal{U}$ be a covering of X. The infinite star of $x \in X$ and $\mathcal{U}$ in X is denoted by $st^{\infty}(x, \mathcal{U})$. Let $C \subseteq X$.

Theorem 4.1. Set C is chain connected in X, if and only if $C \subseteq st^{\infty}(x, \mathcal{U})$ for every $x \in C$ and every covering $\mathcal{U}$ of X.■

Corollary 4.2. Space X is connected, if and only if $X = st^{\infty}(x, \mathcal{U})$ for every $x \in X$ and every covering $\mathcal{U}$ of X.■

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In the next theorem we will give a chain connectedness criterion using continuous function.

Theorem 4.3. X is chain connected, if and only if every continuous function $f : X \to \{0,1\}$ is constant.■

Corollary 4.4. If $f : X \to [0,1]$ is a continuous function such that $f(A) = \{0\}$ and $f(B) = \{1\}$, then sets A and B are separated.■

The reverse claim does not have to be valid.

Example 4.5. The space $X = [-1,1]$ is connected, and hence, chain connected. Let $A = [-1,0)$ and $B = (0,1]$. Then sets A and B are separated, but there is no continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

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Corollary 4.6. Sets A and B are chain separated in $A \cup B$, if and only if the function $f : A \cup B \to \{0,1\}$ ($f : A \cup B \to [0,1]$), such that $f(A) = \{0\}$ and $f(B) = \{1\}$, is continuous. ■

Two nonempty sets A and B are functionally separated in X if there exists a continuous function $f : X \to \{0,1\}$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Therefore, A and B are chain separated sets in $A \cup B$, if and only if A and B are functionally separated in $A \cup B$.

Now we will define a relation of chain connectedness in X.

Let X be a topological space and $x, y \in X$.

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Definition 4.7. Element x is **chain related** to y in X, and we denote it by $x \underset{X}{\sim} y$ or just $x \sim y$, if for every covering $\mathcal{U}$ of X in X there exists a chain in $\mathcal{U}$ that connects x and y.

The chain relation in a topological space is an equivalence relation, and it depends on the set X and the topology τ of X.

Let X be a topological space and $C \subseteq X$.

Remark 4.8. A set C is chain connected in X if and only if for every $x, y \in C$, $x \underset{X}{\sim} y$. ■

Remark 4.9. If a set C is connected, then for every $x, y \in C$, $x \underset{X}{\sim} y$. ■

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Remark 4.10. A space X is connected (chain connected in X) if and only if for every $x, y \in X$, $x \underset{X}{\sim} y$.■

Remark 4.11. If two sets A and B are chain separated in X then for every $x \in A$ and $y \in B$, $x \underset{X}{\not\sim} y$.■

Remark 4.12. Let $x, y \in X$. Then $x \underset{X}{\sim} y$, if and only if X cannot be represented as a union of two nonempty separated sets A and B that contain x and y, respectively.■

Remark 4.13. Let $x, y \in X$. If $x \underset{X}{\sim} y$ and $f : X \to Y$ is a continuous function, then $f(x) \underset{f(X)}{\sim} f(y)$.

5. CHAIN CONNECTED COMPONENTS

Analogous to the component of connectedness, we will define a component of chain connectedness.

Let X be a topological space and $x \in Y \subseteq X$.

Definition 5.1. The **chain connected component** of the point x of Y in X, denoted by $V_{YX}(x)$, is the biggest chain connected subset of Y in X that contains x.

Theorem 5.2. The chain connected component $V_{YX}(x)$ of the point x of Y in X is the set of all points $y \in Y$ such that for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y.

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Proposition 5.3. The set of all chain connected subsets of Y in X consist of all chain connected components of Y in X and their subsets.■

Proposition 5.4. Let $x, y \in Y$. If $y \in V_{YX}(x)$, then

$$V_{YX}(x) = V_{YX}(y).■$$

Theorem 5.5. Let $x, y \in Y$. If $V_{YX}(x) \neq V_{YX}(y)$, then

$$V_{YX}(x) \cap V_{YX}(y) = \varnothing.$$

The chain relation decomposes the space into classes that are chain connected components.

The next proposition shows that every chain connected component of Y in X is a union of chain connected components of Y in Y.

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Proposition 5.6. For every $x \in Y$,

$$V_{YY}(x) \subseteq V_{YX}(x) = \bigcup_{y \in V_{YX}(x)} V_{YY}(y).$$

Proposition 5.7. For every $x \in Y$, $V_{YX}(x) = Y \cap V_{XX}(x)$. Each chain connected component of X in X contains at most one chain connected component of Y in X.■

Proposition 5.8. The chain connected components of X are closed sets, i.e. for every $x \in X$, $V(x) = \overline{V(x)}$.■

Proposition 5.9. Let $x \in X$ and $C(x)$ be a connected component of X. Then $C(x) \subseteq V(x)$.■

Theorem 5.9. Quasicomponents and chain connected components in a topological space X coincide, i.e., for every $x \in X$, $Q_X(x) = V_{XX}(x)$.

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The next proposition is the summary of the propositions 5.3, 5.4, and the theorem 5.3.

Proposition 5.10. For every $x \in Y$,

$$Q_Y(x) = V_{YY}(x) \subseteq \bigcup_{y \in V_{YX}(x)} V_{YY}(y) = V_{YX}(x) \subseteq V_{XX}(x) = Q_X(x).$$

So, chain connected components of a set in a topological space are a union of quasicomponents of the set i.e. for every $x \in C$,

$$V_{CX}(x) = \bigcup_{y \in V_{CX}(x)} Q_C(y),$$

and <u>if</u> the set agrees with the space, the chain connected components match with the quasicomponents.

6 WEAKLY CHAIN SEPARATED SETS

Let X be a topological space, and let $A, B \subseteq X$.

Definition 6.1. The nonempty sets A and B are **weakly chain separated in** X, if for every point $x \in A$ and every $y \in B$, there exists a covering $\mathcal{U} = \mathcal{U}(x, y)$ of X in X such that there is no chain in $\mathcal{U}$ that connects x and y.

Proposition 6.2. If A and B are weakly chain separated in X, then any pair of nonempty sets C and D, where $C \subseteq A$ and $D \subseteq B$, are weakly chain separated in X. ■

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Let X be a topological space, let $Y \subseteq X$, and let A and B be nonempty subsets of Y.

Theorem 6.3. If A and B are weakly chain separated in X, then A and B are weakly chain separated in Y ($A \cup B$).

Theorem 6.4. If the sets A and B are chain separated in X, then A and B are weakly chain separated in X. ■

The next example shows that the converse statement does not hold in general.

Example 6.5. Let $A = \{0\}$, $B = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$, and $X = A \cup B$.

The sets A and B are weakly chain separated in X, but A and B are not chain separated in X i.e. A and B are not separated.

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Corollary 6.6. If A and B are separated then A and B are weakly chain separated in $A \cup B$.■

Theorem 6.7. Singletons A and B are weakly chain separated in X if and only if they are chain separated in X.■

Remark 6.8. Two sets A and B are weakly chain separated in X if and only if for every $x \in A$ and $y \in B$, $x \underset{X}{\not\sim} y$.■

The last proposition in case of chain separateness (remark 4.4 in [1]) is valid in one direction.

Corollary 6.9. Let $x, y \in X$. Then $x \underset{X}{\sim} y$, if and only if X cannot be represented as a union of two weakly chain separated sets A and B that contain x and y, respectively.■

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Theorem 6.10. If the function $f : X \to \{0,1\}$, such that $f(A) = \{0\}$ and $f(B) = \{1\}$, is continuous then the sets A and B are chain separated (weakly chain separated) in X. ■

Corollary 6.11. If A and B are functionally separated in X then A and B are chain separated (weakly chain separated) in X.

From the last corollary it follows that if A and B are functionally separated in $A \cup B$ then A and B are weakly chain separated in $A \cup B$. This statement in the case of chain separateness (corollary 4.3 in [1]) is valid in both directions.

Corollary 6.12. If $f : X \to [0,1]$ is a continuous function such that $f(A) = \{0\}$ and $f(B) = \{1\}$, then the sets A and B are chain separated (weakly chain separated) in $f^{-1}(0) \cup f^{-1}(1)$. ■

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From the last corollary it follows that if $f:X\rightarrow[0,1]$ is a continuous function such that $f(A)=\{0\}$ and $f(B)=\{1\}$, then the sets A and B are chain separated (weakly chain separated) in $A\cup B$. The reverse claim does not have to be valid.

Example 6.13. Let A,B,Y,X are spaces from example 2.2 Then sets A and B are weakly chain separated in $Y=A\cup B$, but there is no continuous function $f:X\rightarrow[0,1]$ such that $f(A)=\{0\}$ and $f(B)=\{1\}$.

At the end we mention the criteria for two kind of topological spaces by using the notion of chain.

Let X be a topological space.

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Theorem 6.14. The space X is the discrete if and only if any two disjoint nonempty subsets of X are chain separated in X.

From the last theorem it follows that we can prove the properties of the discrete space by using the notion of chain.

Theorem 6.15. The space X is totally separated if and only if any two disjoint singletons of X are weakly chain separated in X.

At the end of the section we generalise the notion of totally separated space by using the notion of chain, to a set in a topological space.

Let X be a topolgical space and let A be subspace of X.

Definition 6.16. The set A is totally weakly chain separated in X if any two disjoint singletons of A are weakly chain separated in X.

7 STRONGLY CHAIN CONNECTED SET

Let X be a topological space, and $C \subseteq X$.

Definition 7.1. A set $C(X)$ is **strongly chain connected** in X if $C(X)$ cannot be represented as a union of two weakly chain separated sets A and B in X.

Since the property of weak chain connectedness is weaker then the property of chain connectedness, we expect the property of strong chain connectedness to be stronger then the property of chain connectedness. Actually, the next theorem shows that they are equivalent.

Theorem 7.2. The set $C(X)$ is strongly chain connected in X if and only if $C(X)$ is chain connected in X.

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Theorem 7.3. Let $X = A \cup B$, where A and B are weakly chain separated sets in X, and C is a chain connected set in X. Then $C \subseteq A$ or $C \subseteq B$.

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Thank you