Self-regulating processes

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Motivation

1. Modelling phenomena with varying local regularity is important in various applications: finance, geophysics, internet traffic modelling, ...
2. Often, local regularity varies as a function not of time or space but as a function of the value of the process itself.
3. When local regularity is measured in terms of Hölder exponents, this gives rise to *self-regulating processes*.
4. However, when dealing with discontinuous processes, it is necessary to account for an intensity of jumps that would depend on the value of the process. We refer to those processes as *self-regulating processes*.

Motivation

More precisely, we construct jump processes of a self-regulating nature, that is variants on α-stable processes where the stability index α around time t depends on the value of the process at time t.

The construction utilizes the Poisson sum representation of α-stable processes as a sum over a point set in the plane.

Symmetric α-stable Lévy motion

$\{L_\alpha(t), t \geq 0\}(0 < \alpha \leq 2)$, is the stochastic process with stationary independent increments such that $L_\alpha(0) = 0$ almost surely, and $L_\alpha(t) - L_\alpha(s)$ has the distribution of $S_\alpha((t-s)^{1/\alpha}, 0, 0)$, where $S_\alpha(c, \beta, \mu)$ denotes a stable random variable with stability-index α, with scale parameter c, skewness parameter β, and shift μ.

L_α admits a version with càdlàg sample paths. It may be represented as:

$$L_\alpha(t) = C_\alpha \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} 1_{(0,t]}(\mathsf{X}) \mathsf{Y}^{\langle -1/\alpha \rangle} \tag{2.1}$$

where C_α is a normalising constant, Π is a Poisson point process on $\mathbb{R}^+ \times \mathbb{R}$ with plane Lebesgue measure $\mathcal{L}^2$ as mean measure, and

$$r^{\langle s \rangle} = \mathrm{sign}(r)|r|^s, \quad r \in \mathbb{R}, s \in \mathbb{R}.$$

We are interested in processes where the intensity of jumps α varies. The simplest version of such a process is the *multistable symmetric Lévy motion* $\{M_\alpha(t), t \geq 0\}$ with following representation:

$$M_\alpha(t) = C_\alpha(t) \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} 1_{(0,t]}(\mathsf{X})\mathsf{Y}^{\langle -1/\alpha(t)\rangle} \tag{2.2}$$

where α is a suitable function from $\mathbb{R}^+$ to $(0, 2)$.

$\{M_\alpha(t), t \geq 0\}$ is *localizable* in the following sense: for each $t > 0$ and $u \in \mathbb{R}$,

$$\frac{M_\alpha(t+ru) - M_\alpha(t)}{r^{1/\alpha(t)}} \overset{\text{dist}}{\to} L_{\alpha(t)}(u)$$

as $r \searrow 0$, where convergence is in distribution with respect to the Skorohod metric.

Thus M 'looks like' $\alpha(t)$-stable process near t.

With multistable motion, the local stability parameter depends on the time t.

Our aim is to construct a process Z where the local stability parameter at time t depends instead on the value of the process at time t: for suitable $\alpha : \mathbb{R} \to (0,2)$, $Z(t)$ would be localizable in the sense that for all $t \in [t_0, t_1)$ and $u > 0$

$$\left. \frac{Z(t+ru) - Z(t)}{r^{1/\alpha(Z(t))}} \right| \mathcal{F}_t \overset{\text{dist}}{\to} L^0_{\alpha(Z(t))}(u)$$

as $r \searrow 0$, where $\mathcal{F}_t$ indicates conditioning on the process up to time t.

We first deal with the case where the local jump intensity function ranges in $(0, 1)$ and start by a deterministic construction.

Given a countable discrete point set Π in the plane we define real valued functions f on an interval $[t_0, t_1)$ such that $f(t)$ 'jumps' when $t = x$ for each $(x, y) \in \Pi$, the magnitude of the jump depending both on y and on the value of $\lim_{t \nearrow x} f(t)$.

For $t_0 < t_1$ let $D[t_0, t_1)$ denote the càdlàg functions on $[t_0, t_1)$. The space $D[t_0, t_1)$ is complete under the supremum norm $\| \cdot \|_\infty$.

Fix $0 < a < b < 1$. Let $\alpha : \mathbb{R} \to [a, b]$ be continuously differentiable with bounded derivative and let $\Pi \subset (t_0, t_1) \times \mathbb{R}$ be a set of points such that

$$\sum_{(x,y)\in\Pi} |y|^{-1/b'} < \infty \tag{3.3}$$

for some $b < b' < 1$.

Let $a_0 \in \mathbb{R}$. Define K on $D[t_0, t_1)$ by

$$K(f)(t) = a_0 + \sum_{(x,y)\in\Pi} 1_{(t_0,t]}(x) y^{\langle -1/\alpha(f(x_-))\rangle} \qquad (t_0 \le t < t_1), \quad (3.4)$$

where the sum is absolutely convergent by (3.3).

Lemma

The operator K maps $D[t_0, t_1)$ into itself.

Theorem
With a_0, α and Π as above, there exists a unique $f \in D[t_0, t_1)$ such that

$$f(t) = a_0 + \sum_{(x,y)\in\Pi} 1_{(t_0,t]}(x) y^{\langle -1/\alpha(f(x_-))\rangle} \qquad (t_0 \leq t < t_1). \tag{3.5}$$

In particular $f(t_0) = a_0$. Moreover, for each s and t with $t_0 \leq s < t < t_1$, $f(t)$ is completely determined given $f(s)$ and the points of the set $\Pi \cap ((s,t] \times \mathbb{R})$.

Idea of proof:

We would like to use that K is a contracting operator on $D[t_0, t_1)$ and apply Banach's contraction theorem. However, K is contracting only if the value of $|y|$ is not too small at points $(x, y) \in \Pi$.

We use that K is contracting in intervals where $|y|$ is not too small and incorporate the jumps at these points 'by hand'.

The function f may be approximated as follows: define a sequence of functions f_n $(n \in \mathbb{N})$ by restricting the sums to points with $|y| \leq n$, that is :

$$f_n(t) = a_0 + \sum_{(x,y)\in\Pi\,:\,|y|\leq n} 1_{(t_0,t]}(x) y^{\langle -1/\alpha(f_n(x_-))\rangle} \tag{3.6}$$

for $t_0 \leq t < t_1$. Then $f_n \in D[t_0, t_1)$ is uniquely defined as a sum over a finite set of points and is piecewise constant, so it may be evaluated using a finite number of inductive steps.

Theorem

$\{f_n\}$ is a Cauchy sequence in $(D[t_0, t_1), \|\cdot\|_\infty)$. Moreover, $f_n \to f$ in $\|\cdot\|_\infty$ and we have:

$$\|f_n - f\|_\infty \leq \exp\left(M \sum_{(x,y)\in\Pi, |y|\leq n} |y|^{-1/(a,b)}\right) \sum_{(x,y)\in\Pi : |y|>n} |y|^{-1/b},$$

where

$$M = \sup_{\xi\in\mathbb{R}} \frac{|\alpha'(\xi)|}{\alpha(\xi)^2}.$$

Idea of proof:

The main difficulty with the sequence of functions $f_n(t)$ is that when, as n increases, a new point (x, y) enters the sum then, for all existing (x', y') with $x' > x$ and smaller $|y'|$, the summands $y'^{\langle -1/\alpha(f(x'_-))\rangle}$ will change, leading to a change in $f_n(t)$ for $t > x$ that is amplified as t increases past larger x with $(x, y) \in \Pi$.

We deal with this difficulty by carefully controlling these changes.

Going back to the random case, we have the following result:

Theorem

Let $\Pi \subset (t_0, t_1) \times \mathbb{R}$ be a Poisson point process with $\mathcal{L}^2$ as mean measure. Then there exists a Markov process Z on $[t_0, t_1)$ such that, almost surely, the sample paths are in $D[t_0, t_1)$ with $Z(t_0) = a_0$ and

$$Z(t) = a_0 + \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} 1_{(t_0,t]}(\mathsf{X})\, \mathsf{Y}^{\langle -1/\alpha(Z(\mathsf{X}_-))\rangle} \qquad (t_0 \leq t < t_1). \quad (3.7)$$

Writing

$$Z_n(t) = a_0 + \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi : |\mathsf{Y}| \leq n} 1_{(t_0,t]}(\mathsf{X})\, \mathsf{Y}^{\langle -1/\alpha(Z_n(\mathsf{X}_-))\rangle} \qquad (t_0 \leq t < t_1) \quad (3.8)$$

then almost surely, $\|Z_n - Z\|_\infty \to 0$ as $n \to \infty$.

We investigate the local properties of Z.

Proposition

Given $0 < \epsilon < 1/b$, for each $t \in [t_0, t_1)$ there exists almost surely a random $C > 0$ such that for all $0 < h < t_1 - t$,

$$|Z(t+h) - Z(t)| \leq C h^{1/\alpha(Z(t)) - \epsilon}. \tag{3.9}$$

Idea of proof:

The main ingredient is a comparison with the α-stable subordinator S_α, for constant $0 < \alpha < 1$:

$$S_\alpha(t) \;:=\; \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} 1_{(0,t]}(\mathsf{X})\,|\mathsf{Y}|^{-1/\alpha}.$$

S_α is a self-similar process with stationary increments such that for all $0 < \epsilon < 1/\alpha$ there is almost surely a random constant $C < \infty$ such that

$$S_\alpha(t) \;\le\; Ct^{(1/\alpha)-\epsilon} \qquad (t \ge 0). \tag{3.10}$$

Theorem

Z is right-localizable at each $t \in [t_0, t_1)$, in the sense that

$$\left.\frac{Z(t+ru)-Z(t)}{r^{1/\alpha(Z(t))}}\right| \mathcal{F}_t \overset{\text{dist}}{\to} L^0_{\alpha(Z(t))}(u) \qquad (3.11)$$

as $r \searrow 0$, where convergence is in distribution with respect to $(D[0,t_1), \rho_S)$, with ρ_S the Skorohod metric.

Idea of proof:

We compare $Z(t+ru)-Z(t)$ and $L^0_{\alpha(Z(t))}(t+ru)-L^0_{\alpha(Z(t))}(t)$, where

$$L^0_\alpha(t) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} 1_{(0,t]}(\mathsf{X})\mathsf{Y}^{\langle -1/\alpha\rangle} \qquad (t\geq 0). \tag{3.12}$$

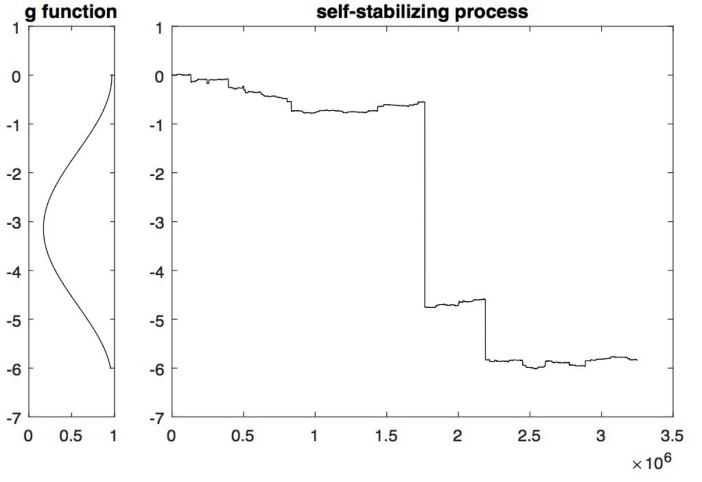

Figure: left : self-stabilizing function $\alpha(z) = 0.57 + 0.4\cos(z)$. Right: corresponding realization of a self-stabilizing process.

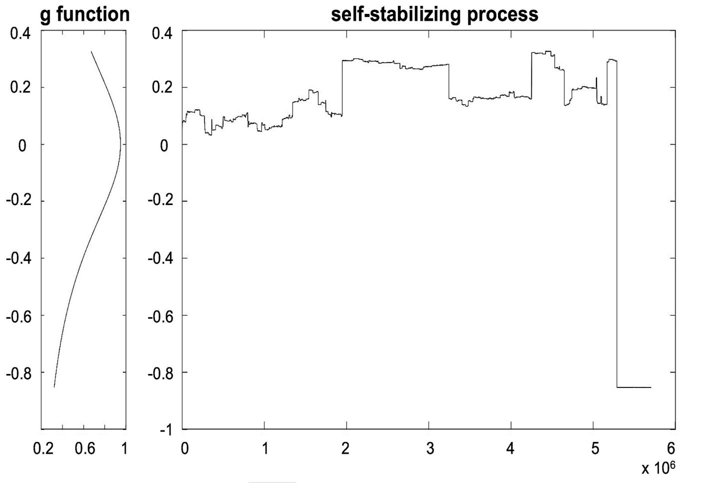

Figure: left : self-stabilizing function $\alpha(z) = 0.15 + \frac{0.8}{1+5z^2}$. Right: corresponding realization of a self-stabilizing process.

When $\alpha : \mathbb{R}^+ \to (0, 1)$, we started by showing that there exists a deterministic function $f \in D[t_0, t_1)$ satisfying the relation

$$f(t) = a_0 + \sum_{(x,y) \in \Pi} 1_{(t_0,t]}(x) y^{\langle -1/\alpha(f(x_-)) \rangle}$$

for a fixed point set Π, and then randomising to get a random function Z such that

$$Z(t) = a_0 + \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} 1_{(t_0,t]}(\mathsf{X}) \, \mathsf{Y}^{\langle -1/\alpha(Z(\mathsf{X}_-)) \rangle} \qquad (t_0 \le t < t_1).$$

However, this approach depends on the infinite sums being absolutely convergent, which need not be the case if $\alpha(t) \geq 1$ for some t.

The general case is more involved.

We show first that for a *fixed* point set $\Pi^+ \subset (t_0, t_1) \times \mathbb{R}^+$ and independent random 'signs' $S(x, y) = \pm 1$ there exists almost surely a random function $Z^p \in D[t_0, t_1)$ satisfying

$$Z^p(t) = a_0 + \sum_{(x,y) \in \Pi^+} 1_{(t_0,t]}(x) S(x, y) y^{-1/\alpha(Z^p(x_-))} \qquad (t_0 \le t < t_1).$$

To achieve this we work with partial sums

$$Z^p_n(t) = a_0 + \sum_{(x,y) \in \Pi^+ : |y| \le n} 1_{(t_0,t]}(x) S(x, y) y^{-1/\alpha(Z^p_n(x_-))} \qquad (t_0 \le t < t_1)$$

and show that the limit as $n \to \infty$ exists in a norm given by $\mathbb{E}(\| \cdot \|^2_\infty)^{1/2}$, where $\mathbb{E}$ denotes expectation.

Theorem

$$\mathbb{E}\left(\|Z_n^p - Z^p\|_\infty^2\right) \leq 4 \prod_{(x,y)\in\Pi^+} \left(1 + M^2 y^{-2/(a,b)}\right) \sum_{(x,y)\in\Pi^+ : y>n} y^{-2/b} \to 0. \tag{4.13}$$

Proposition

Let $Z \in \mathcal{D}$ be the random function given by Theorem 4.1 and let $t \in [t_0, t_1)$. Suppose that for some $\beta > 0$,

$$\sum_{(x,y)\in\Pi^+ : t<x\leq t+h} y^{-2/\alpha(Z(t))} = O(h^\beta) \qquad (0 < h < t_1 - t). \quad (4.14)$$

Then, conditional on $\mathcal{F}_t$, given $0 < \epsilon < \beta$ there exist almost surely a random number $C_1 < \infty$ such that for all $0 \leq h < t_1 - t$,

$$|Z(t+h) - Z(t)| \leq C_1 h^{(\beta-\epsilon)/2}. \quad (4.15)$$

We now randomise the construction further by taking Π^+ to be a Poisson point process on $(t_0, t_1) \times \mathbb{R}^+$ with mean measure $2\mathcal{L}^2$.

The key idea is that the distribution of the point sets $\{(\mathsf{X}, \mathsf{Y}) \in \Pi\}$ where $\Pi \subset (t_0, t_1) \times \mathbb{R}$ is a Poisson point process with plane Lebesgue measure $\mathcal{L}^2$ as mean measure, is identical to that of $\{(\mathsf{X}, S(\mathsf{X}, \mathsf{Y})\mathsf{Y}) : (\mathsf{X}, \mathsf{Y}) \in \Pi_2^+, S(\mathsf{X}, \mathsf{Y}) = \pm 1\}$, where Π_2^+ is a Poisson point process on $(t_0, t_1) \times \mathbb{R}^+$ with double Lebesgue measure $2\mathcal{L}^2$ as mean measure and with the $S(\mathsf{X}, \mathsf{Y})$ independently taking the values ± 1 with equal probability $\frac{1}{2}$ for each $(\mathsf{X}, \mathsf{Y}) \in \Pi_2^+$ (this follows from the superposition property of Poisson processes).

Hence Π can be realised by first sampling (X, Y) from Π_2^+ and then assigning random signs to the Y coordinates.

Theorem

Let $\Pi \subset (t_0, t_1) \times \mathbb{R}$ be a Poisson point process with mean measure $\mathcal{L}^2$, let $\alpha : \mathbb{R} \to [a, b]$ where $0 < a < b < 2$ and let $a_0 \in \mathbb{R}$. Then there exists $Z \in \mathcal{D}$ such that $\lim_{n\to\infty} \mathbb{E}\big(\|Z_n - Z\|_\infty^2\big) = 0$ where Z_n is defined as

$$Z_n(t) = a_0 + \sum_{(\mathsf{X},\mathsf{Y})\in\Pi:|\mathsf{Y}|\le n} 1_{(t_0,t]}(\mathsf{X})\,\mathsf{Y}^{\langle -1/\alpha(Z_n(\mathsf{X}_-))\rangle} \qquad (t_0 \le t < t_1).$$

Theorem

Let $y_0 > 0$ and let Π be a Poisson point process on $(t_0, t_1) \times (-\infty, -y_0] \cup [y_0, \infty)$ with mean measure $\mathcal{L}^2$ restricted to this domain. Then,

$$\mathbb{E}(\|Z_n - Z\|_\infty^2) \leq \frac{8b(t_1 - t_0)}{2 - b} \exp\left(2M^2(t_1 - t_0) \int_{y_0}^{\infty} y^{-2/(a,b)}\, dy\right) n^{-(2-b)/b}$$

Proposition

Conditional on $\mathcal{F}_t$, given $0 < \epsilon < 1/b$, there exist almost surely random numbers $C_1, C_2 < \infty$ such that for all $0 \leq h < t_1 - t$,

$$|Z(t+h) - Z(t)| \leq C_1 h^{1/\alpha(Z(t)) - \epsilon}$$

and

$$\left|\left(Z(t+h) - Z(t)\right) - \left(L^0_{\alpha(t)}(t+h) - L^0_{\alpha(t)}(t)\right)\right| \leq C_2 h^{1/\alpha(Z(t)) + 1/b - \epsilon},$$

where $L^0_{\alpha(t)}$ is the $\alpha(t)$-stable process defined using the same realisations of Π as Z.

Theorem

Let $t \in [t_0, t_1)$. Then, conditional on $\mathcal{F}_t$, almost surely Z is right-localisable at t, in the sense that

$$\left.\frac{Z(t+ru) - Z(t)}{r^{1/\alpha(Z(t))}}\right| \mathcal{F}_t \overset{\text{dist}}{\to} L^0_{\alpha(Z(t))}(u) \qquad (0 \le u \le 1)$$

as $r \searrow 0$, where convergence is in distribution with respect to $(D[0,1], \rho_S)$, with ρ_S the Skorohod metric.