Topology of symplectic fillings of contact 3-manifolds

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8ECM

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At the turn of the century
there were two groundbreaking results about
symplectic and contact topology
in low-dimensions.

- **Donaldson**'s result about the existence of **Lefschetz pencils** on closed **symplectic** 4-manifolds, and
- **Giroux**'s correspondence between **open books** and **contact structures** on closed 3-manifolds.

Topological characterization of symplectic 4*-manifolds*

Lefschetz fibrations

Suppose that X and Σ are compact, oriented and smooth manifolds of dimensions **four** and **two**, respectively, *possibly with nonempty boundaries.*

Definition : A **Lefschetz fibration** $\pi\colon X \to \Sigma$ is a submersion except for finitely many points $\{p_1, \ldots, p_k\}$ in the interior of X, such that around each p_i and $\pi(p_i)$, there are orientation-preserving complex charts, on which π is of the form

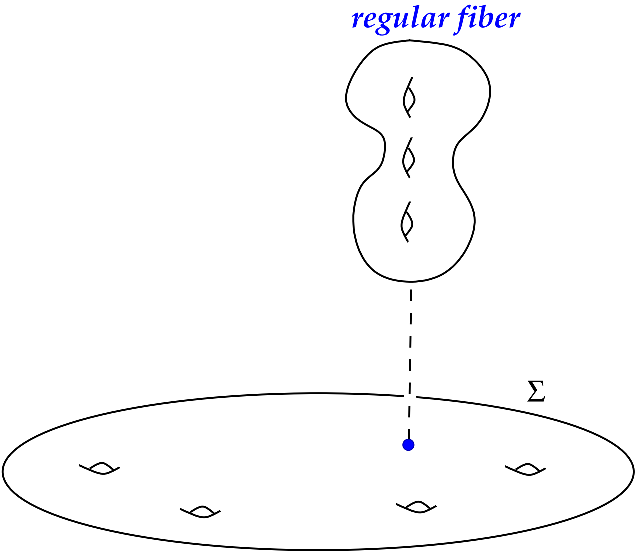
$$\pi(z_1, z_2) = z_1^2 + z_2^2.$$

Lefschetz critical points can be viewed as complex analogs of **Morse critical points**, and they correspond to 2-handles. As a result one obtains a **handle decomposition** of X.

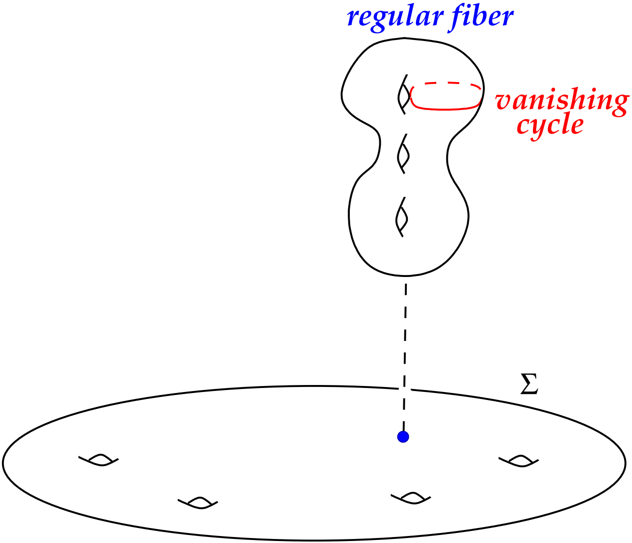
Lefschetz fibrations can also be described *combinatorially* by means of their **monodromy.**

Locally :

The fiber of the map $(z_1, z_2) \to z_1^2 + z_2^2$ above $0 \neq t \in \mathbb{C}$ is smooth *(topologically an annulus)*, while the fiber above the origin has a **transverse double point** ***(nodal singularity)*** and is obtained from the nearby fibers by *collapsing an embedded simple closed loop* called the **vanishing cycle**.



regular fiber
Σ


regular fiber
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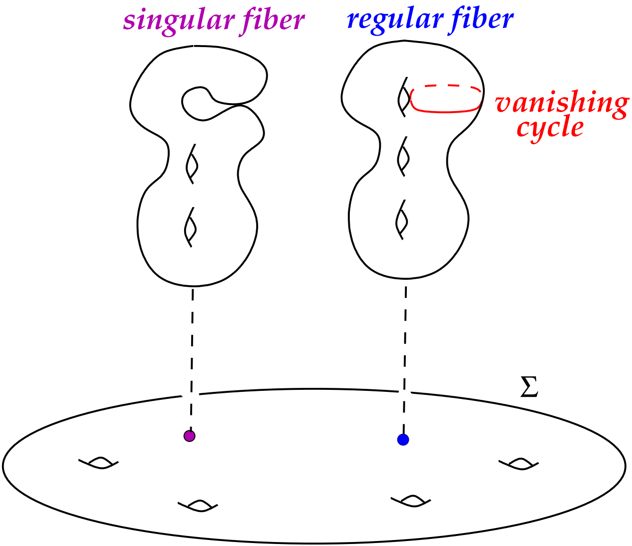
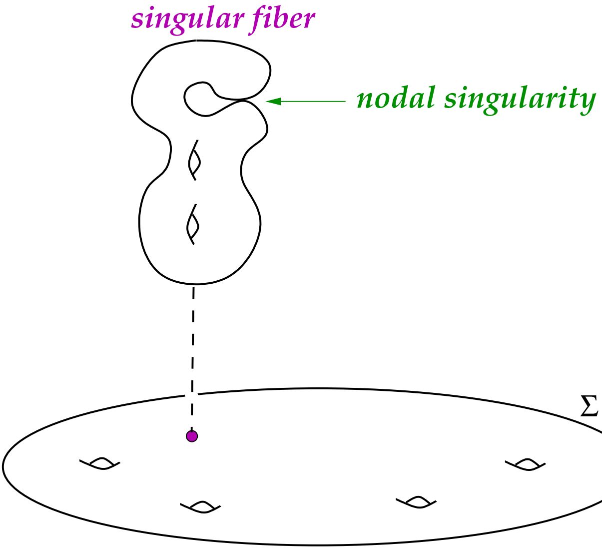
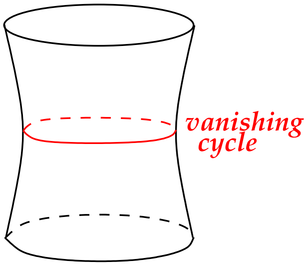
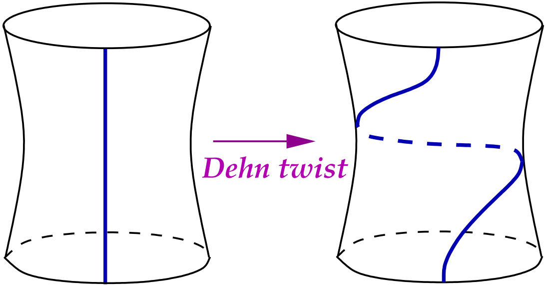

singular fiber
regular fiber
vanishing cycle
Σ



singular fiber
nodal singularity
Σ

Monodromy : Given a Lefschetz fibration $\pi\colon X \to \Sigma$. Fix a **loop** in Σ enclosing a single critical value, whose critical fiber has a single node. Then π restricts to surface fibration over this **loop** whose monodromy *(a diffeomorphism of the fiber)* is given by the **right-handed Dehn twist** about the **vanishing cycle.**





For the purposes of this talk, we restrict our attention to the cases where either

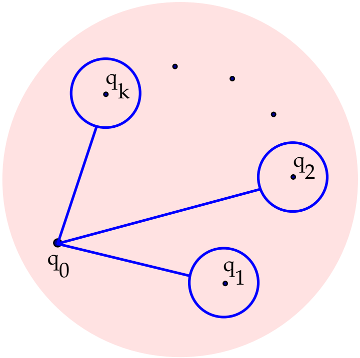
- Σ is the 2-sphere S^2, and the fibres are **closed** surfaces (thus $\partial X = \emptyset$), or
- Σ is the 2-disk D^2, and the fibres have **nonempty boundaries** (thus $\partial X \neq \emptyset$).

We also assume that

(i) each singular fiber carries a **unique singularity** and

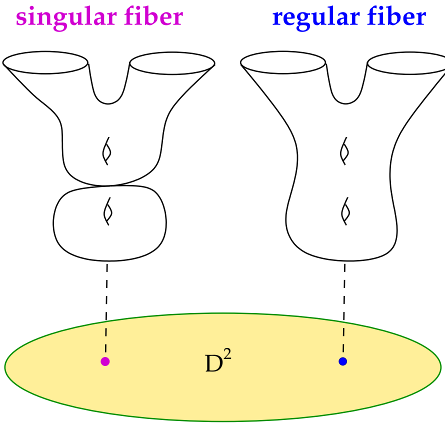
(ii) there are ***no homotopically trivial vanishing cycles***.

First case, $\Sigma = S^2$, $\partial X = \emptyset$: If $q_1, \ldots, q_k \in S^2$ are the **critical** values of a **genus g** Lefschetz fibration $\pi\colon X \to S^2$, then by fixing a point q_0 in S^2, we can characterize $\pi\colon X \to S^2$ by means of its **monodromy** morphism $\psi\colon \pi_1(S^2 - \{q_1, \ldots, q_k\}) \to Map_g$, where Map_g is the **mapping class group** of an oriented closed genus g surface.



Upshot : A Lefschetz fibration $\pi\colon X \to S^2$ is **characterized** by a **positive Dehn twist factorization of the identity element** in Map_g (*uniquely determined up to some natural equivalences*).

Second case, $\Sigma = D^2$, $\partial X \neq \emptyset$: This case is simpler, since the global monodromy is a product of **positive Dehn twists** in $Map_{g,r}$ (*with no other constraints*).



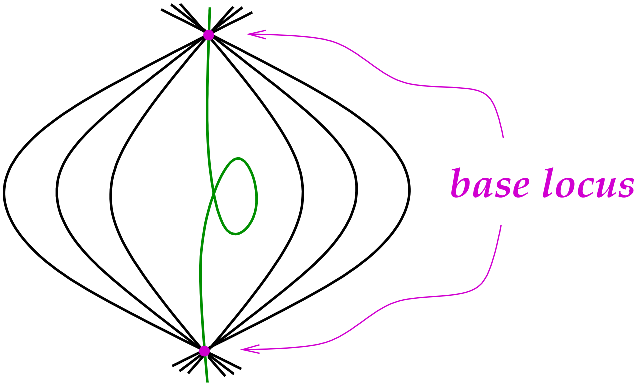
In this case, ∂X inherits a natural **open book decomposition**, which we will discuss in details later.

Lefschetz pencils

Definition : A **Lefschetz pencil** on a **closed** and oriented 4-manifold X is a map $\pi\colon X - \{b_1, \ldots, b_n\} \to S^2$, submersive except for a finite set $\{p_1, \ldots, p_k\}$, conforming to **local models**

(i) $(z_1, z_2) \to z_1/z_2$ near each b_i and

(ii)$(z_1, z_2) \to z_1^2 + z_2^2$ near each p_j.

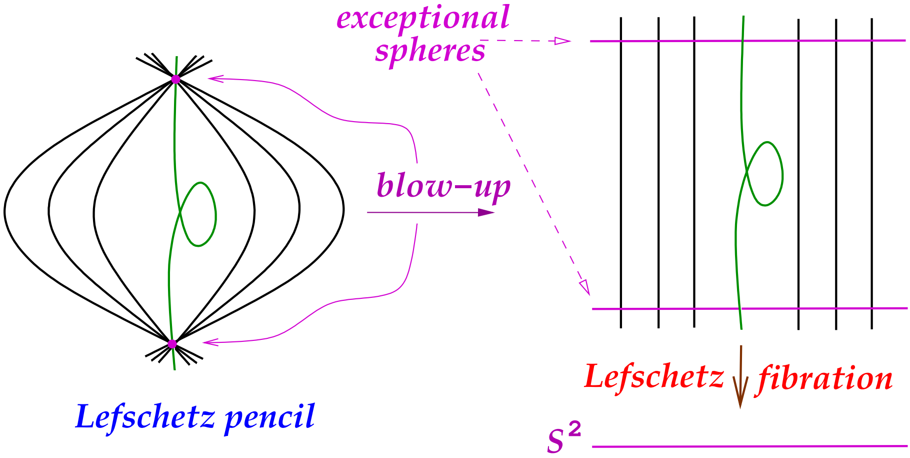


Lefschetz pencil

By **blowing up** X at the **base points** $\{b_1, \ldots, b_n\}$, we obtain a **Lefschetz fibration**

$$X \# n\overline{\mathbb{C}P^2} \to S^2$$

with n **disjoint sections**, which are the exceptional spheres in the blow-up.


exceptional spheres
blow−up
Lefschetz
fibration
Lefschetz pencil
S^2

In 1924, **Lefschetz** showed that **projective surfaces** admit Lefschetz pencils —which was extended by Donaldson, to closed **symplectic** 4-manifolds (i.e. those admitting **closed non-degenerate 2-forms**).

Theorem (Donaldson 1999)

Any closed symplectic 4-manifold admits a Lefschetz pencil.

Conversely, **Gompf' 99** (*generalizing a similar result of* **Thurston' 76** *on surface bundle over surfaces*) showed that if $\pi : X \to \Sigma$ is a Lefschetz fibration for which the fiber represents a non-torsion homology class, then X admits a symplectic structure with **symplectic fibers.**

The hypothesis is satisfied if the fiber genus is not equal to one.

As a corollary, he showed that any closed 4-manifold which admits a Lefschetz pencil, is **symplectic**.

This **topological characterization** of symplectic 4-manifolds has lead to a renewed interest in **Lefschetz pencils/fibrations** and hundreds of papers have been devoted to the study of various aspects and *(generalizations of)* Lefschetz fibrations. Here is one of the earlier results :

Theorem [**O. & Stipsicz' 00**, independently **Smith' 00**]
There are infinitely many (pairwise non-homeomorphic) closed 4-manifolds, which admit genus two Lefschetz fibrations over S^2 but do not carry complex structure with either orientation.

These examples are obtained by **fiber sums** of genus two Lefschetz fibrations $S^2 \times T^2 \# 4\overline{\mathbb{C}P^2} \to S^2$ of **Matsumoto' 95**.

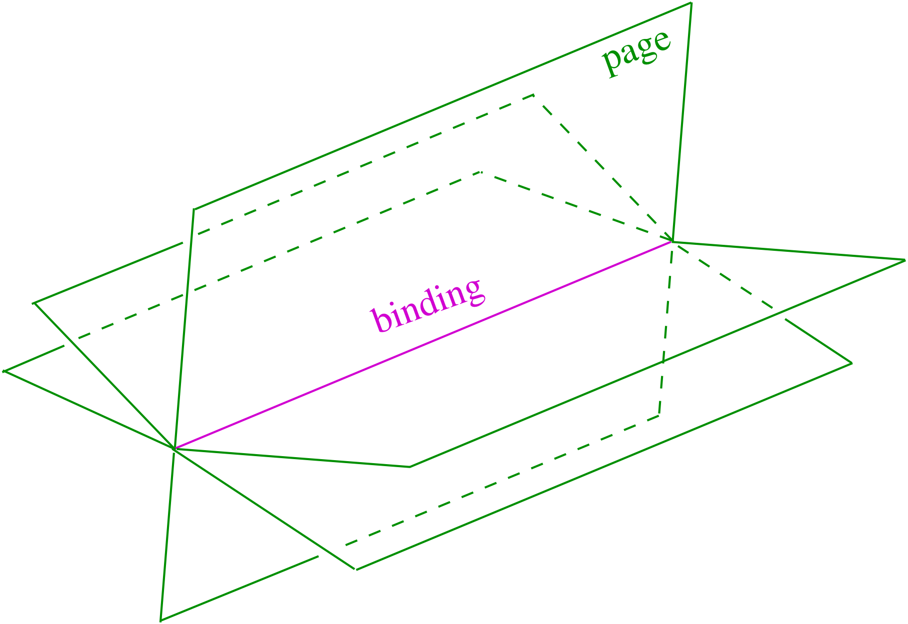
$\Longrightarrow$ **fiber sums** of **holomorphic** Lefschetz fibrations is *not necessarily holomorphic.*

Topological characterization of contact 3*-manifolds*

Open book decompositions

Definition : An **open book decomposition** of a closed and oriented 3-manifold Y is a pair (B, π) consisting of an **oriented link** $B \subset Y$, and a **locally-trivial fibration** $\pi \colon \; Y - B \to S^1$ such that B has a trivial tubular neighborhood $B \times D^2$ in which π is given by the angular coordinate in the D^2-factor.

Here B is called the **binding** and the closure of each fiber, which is a Seifert surface for B, is called a **page.**



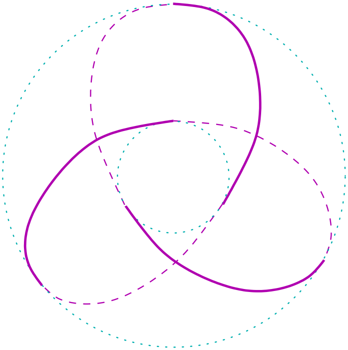
I am an open book !

Example : (**Milnor's fibration**)

Consider the polynomial $f\colon \mathbb{C}^2 \to \mathbb{C}$ given by $f(z_1, z_2) = z_1^p + z_2^q$, where $p, q \geq 2$ are relatively prime. Then $B = f^{-1}(0) \cap S^3$ is the $(p,q)-$torus knot in S^3 whose complement fibers over S^1 :

$$\pi\colon S^3 - B \to S^1 := \frac{f(z_1, z_2)}{|f(z_1, z_2)|}$$

(B, π) is an **open book** for S^3 with connected binding.



(2, 3)-torus knot
(**trefoil**)

The topology of an open book is determined by the topology of its page and its monodromy : Choose a vector field which is **transverse** to the pages and meridional near the binding. The isotopy class of the **first return map** on a fixed page is the **monodromy** of the open book.

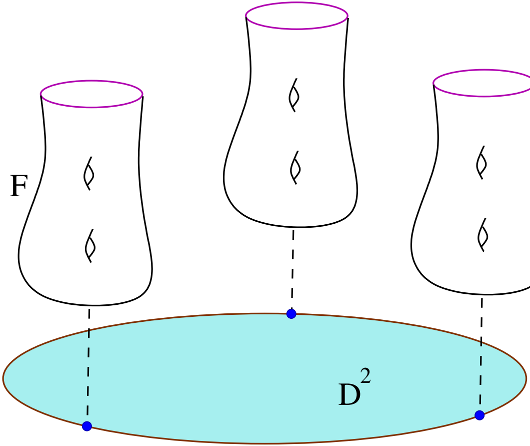
Suppose that $\pi : X \to D^2$ is a Lefschetz fibration such that the regular fiber F has **nonempty boundary** ∂F. Then ∂X is the union of two pieces :

- the **horizontal** boundary, $\partial F \times D^2$ and
- the **vertical** boundary, $\pi^{-1}(\partial D^2)$,

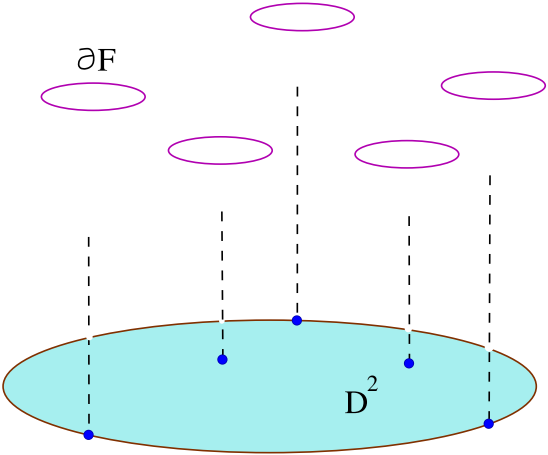
glued together along the tori $\partial F \times \partial D^2$.

It follows that ∂X inherits a natural **open book**, whose page is the fiber F and whose monodromy coincides with the monodromy of the Lefschetz fibration $\pi : X \to D^2$.

The **vertical** boundary : $\pi^{-1}(\partial D^2)$



The **horizontal** boundary : $\partial F \times D^2$



Contact 3-manifolds

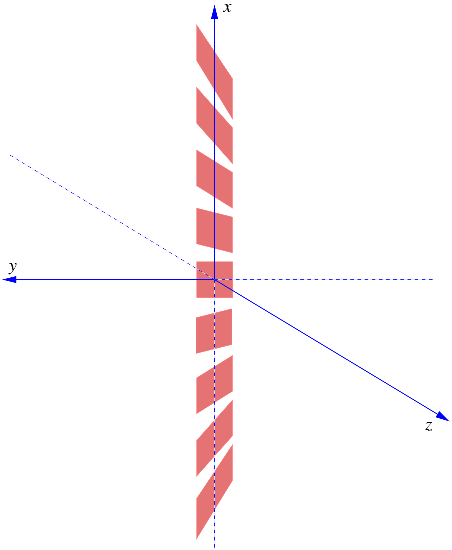
A differential 1-form α on a 3-manifold Y is a called a **contact form** if $\alpha \wedge d\alpha$ is a volume form.

A 2-dimensional distribution ξ in TY is called a **contact structure** if it can be given as the **kernel** of a contact form α.

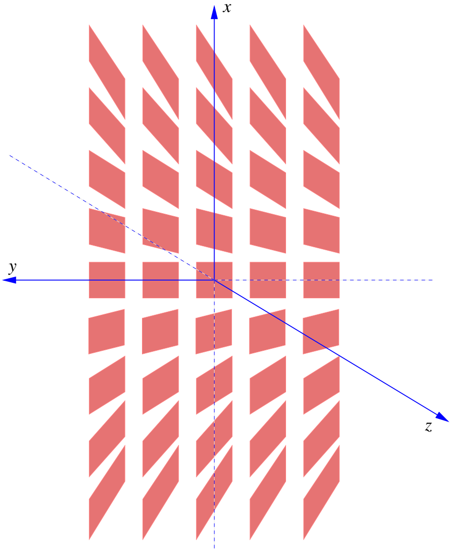
The pair (Y, ξ) is called a ***contact 3-manifold.***

Darboux's Theorem : Any point in a contact 3-manifold has a neighborhood isomorphic to a neighborhood of the origin in the **standard** contact structure $\xi = \ker(dz + xdy)$ in $\mathbf{R}^3$.

The standard contact structure $\xi = \ker(dz + xdy)$ in $\mathbb{R}^3$



The standard contact structure $\xi = \ker(dz + xdy)$ in $\mathbb{R}^3$



Open books and contact structures on 3-manifolds

- **Alexander' 23 :** Every closed oriented 3-manifold has an open book decomposition.
- **Martinet' 71 :** Every closed oriented 3-manifold has a contact structure.
- **Thurston & Winkelnkemper' 75** constructed contact forms on closed 3-manifolds using open books, *giving an alternate proof of Martinet's theorem.*

Definition : We say that a contact structure ξ on a 3-manifold Y is **supported** by an open book (B, π) if ξ can be given by a contact form α such that $\alpha(B) > 0$ and $d\alpha > 0$ on every page.

Thurston & Winkelnkemper' 75 : *Every open book on a closed oriented 3-manifold supports a contact structure.*

The converse (i.e. *every contact structure on* Y *is supported by an open book*) was proven by **Giroux' 00**. In fact he proved the following theorem.

Theorem (Giroux's correspondence)

On a closed oriented 3*-manifold, there is a* **one-to-one correspondence** *between the set of isotopy classes of contact structures and the open books up to positive stabilization.*

Stein manifolds

Definition : A **Stein manifold** is an affine complex manifold, i.e., a complex manifold that admits a proper holomorphic embedding into some $\mathbb{C}^N$.

If $\phi\colon\ X \to \mathbb{R}$ is a smooth function on a complex manifold (X, J), then $\omega_\phi := -d(d\phi \circ J)$ is a 2-form. The map $\phi\colon\ X \to \mathbb{R}$ is called ***J*-convex** (aka *strictly plurisubharmonic*) if $\omega_\phi(u, Ju) > 0$ for all nonzero vectors $u \in TX$.

It follows that ω_ϕ is an **exact symplectic** form on X.

Grauert's characterization : A **complex** manifold (X, J) is **Stein** if and only if it admits a proper J-convex function $\phi\colon\ X \to [0, \infty)$.

For the purposes of this talk, we now restrict our attention to **Stein surfaces** (**complex 2-dimensional**)

Let (X, J) be a Stein surface. For any J-convex *Morse* function $\phi\colon X \to [0, \infty)$, each **regular level set** Y of ϕ is a contact 3-manifold, where the contact structure is given by the kernel of $\alpha_\phi = -d\phi \circ J$ (or equivalently, by the *complex tangencies* $TY \cap JTY$).

For any regular value c of ϕ, the sublevel set $W = \phi^{-1}([0, c])$ is called a **Stein domain.** We also say that the compact 4-manifold (W, J) is a **Stein filling** of its contact boundary $(\partial W, \xi = \ker \alpha_\phi)$.

Topological characterization of Stein domains
(of complex dimension two)

By the work of **Eliashberg' 90**, and **Gompf' 98**, a **handle decomposition** of a **Stein domain** (W, J) is well-understood :

It consists of

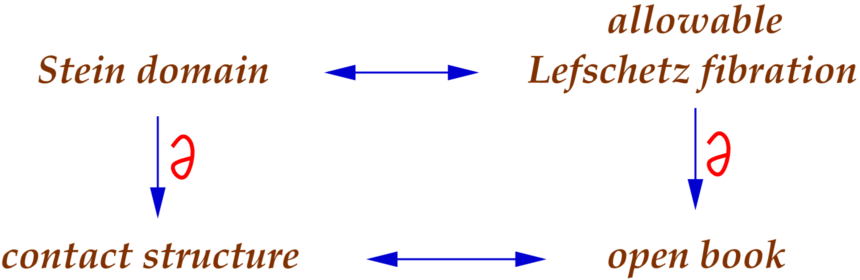
- a 0-handle,
- some 1-handles and
- some 2-handles attached along **Legendrian knots** *(those tangent to the contact planes)* with framing -1 relative to the contact planes.

Theorem [**Akbulut & O.' 01** and **Loi &Piergallini' 01**]
A Stein domain admits an **allowable*** *Lefschetz fibration over* D^2 *and conversely, an allowable Lefschetz fibration over* D^2 *admits a Stein structure.*

***allowable** : the vanishing cycles are **homologically non-trivial.**

<u>**Idea of proof :**</u> **Stein** 2-handle $\Leftrightarrow$ **Lefschetz** 2-handle

Moreover, by slightly modifying the proof of Akbulut and myself, **Plamenevskaya' 04** showed that *the* **contact structure** *induced on the boundary of the Stein domain is supported by the* **open book** *inherited by the Lefschetz fibration.*



A criterion* for Stein fillability : A contact 3-manifold is **Stein fillable** if and only if it admits a supporting open book whose monodromy can be factorized into **positive Dehn twists.**

*This was independently proved by **Giroux**.

An active line of research topic in symplectic/contact topology is to **classify** ***all Stein fillings***, or more generally ***all minimal symplectic fillings*** of a given contact 3-manifold, up to diffeomorphism.

Definition : A compact symplectic 4-manifold (X, ω) is a (strong) **symplectic filling** of a contact 3-manifold (Y, ξ) if $\partial X = Y$ (as oriented manifolds), ω is **exact near the boundary** and its primitive α can be chosen so that $\ker(\alpha|_Y) = \xi$.

A symplectic filling is called **minimal** if it does not contain any symplectically embedded sphere of self-intersection -1.

Every Stein filling is a minimal symplectic filling.

Converse is *not true* as shown by **Ghiggini' 05**, using **Ozsváth-Szabó contact invariants.**

The ***classification*** of **Stein**/**minimal symplectic fillings** of a given contact 3-manifold is difficult in general. Nevertheless, this problem has been solved for many contact 3-manifolds, each of which has finitely many fillings.

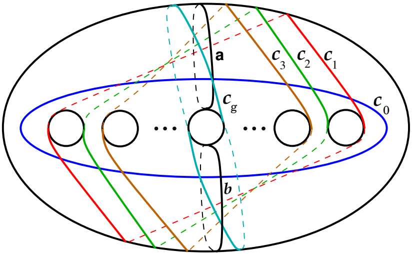
The first example of a contact 3-manifold which admits **infinitely many distinct Stein fillings** was given by **Stipsicz** and myself :

Let Y_g denote the closed 3-manifold, which is the total space of the open book whose page is a genus g surface with connected boundary and whose monodromy is the **square of the boundary Dehn twist.** Let ξ_g denote the contact structure on Y_g supported by this open book.

Theorem [**O. & Stipsicz' 04**]
For each odd integer $g \geq 3$, the contact 3-manifold (Y_g, ξ_g) admits infinitely many pairwise non-homeomorphic Stein fillings.

Outline of proof :

- A positive word in Map_g, for $g \geq 3$ (generalizing **Matsumoto's genus two word**), was discovered by **Cadavid' 01** and **Korkmaz' 01**, independently : For g odd, $(c_0 c_1 c_2 \ldots c_g a^2 b^2)^2 = 1$



- Take (twisted) **fibers sums** of Lefschetz fibrations over S^2
- Remove a section and a fiber, to get Stein fillings of the common contact boundary
- Distinguish the Stein fillings by **torsion in first homology** coming from the twisting.

Baykur & Van Horn-Morris' 15 : There are vast families of contact 3-manifolds each member of which admits infinitely many Stein fillings with **arbitrarily large Euler characteristics** and **arbitrarily small signature.**

Canonical contact structures on the links of isolated complex surface singularities

A fruitful source of Stein fillable contact 3-manifolds is given by the **links of isolated complex surface singularities.**

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an isolated complex surface singularity. Then for a sufficiently small sphere $S_\epsilon^{2N-1} \subset \mathbb{C}^N$ centered at the origin, $Y = X \cap S_\epsilon^{2N-1}$ is a closed, oriented and smooth 3-dimensional manifold, which is called **the link of the singularity.**

If J denotes the complex structure on X, then the complex tangencies $\xi := TY \cap JTY$ is a contact structure on Y— called the **canonical** (aka **Milnor fillable**) contact structure on the singularity link. We refer to (Y, ξ) as the **contact singularity link**, in short.

Note that ξ is determined uniquely, up to isomorphism, by a theorem of **Caubel & Némethi & Popescu-Pampu' 06**.

The **minimal resolution** of the singularity provides a Stein filling of the contact singularity link (Y, ξ) (by the work of **Bogomolov & de Oliveira' 97**)

$\Longrightarrow (Y, \xi)$ is Stein fillable.

Moreover, if the singularity is smoothable, the general fiber X of a **smoothing** is called a **Milnor fiber**, which is a compact smooth 4-manifold such that $\partial X = Y$. Furthermore, X has a natural Stein structure so that it provides a **Stein** *(hence minimal symplectic)* filling of (Y, ξ).

Question : Does there exist a **contact singularity link** which admits Stein (or minimal symplectic) fillings other than the Milnor fibers (and the minimal resolution) ?

No, for **simple and simple elliptic singularities**
Ohta & Ono' 03 + 05

No, for **cyclic quotient singularities**
McDuff' 90 + Christophersen' 91 + Stevens' 91 + Lisca' 08 + Némethi & Popescu-Pampu' 10

No, for **non-cyclic quotient singularities**
Stevens' 93 + Bhupal & Ono' 12 + H. Park & J. Park & Shin & Urzua' 18

Yes, for some **Seifert fibered singularity links**, as shown by **Akhmedov** and myself.

Theorem [**Akhmedov & O.' 14**]
There exists an infinite family of **Seifert fibered contact singularity links** *such that each member of this family admits* **infinitely many exotic*** **Stein fillings***. Moreover,* **none of these Stein fillings are homeomorphic to Milnor fibers.**

***exotic** : homeomorphic but pairwise not diffeomorphic

Our examples are not simply-connected. The first examples of infinitely many exotic **simply-connected** Stein fillings were discovered by **Akhmedov & Etnyre & Mark & Smith' 08**.

Recently, **Plamenevskaya & Starkston' 21** showed that many **rational singularities** admit *(simply-connected)* Stein fillings that are not diffeomorphic to any Milnor fibers.

Theorem [**Akhmedov & O.' 18**]
For any finitely presented group G, there exists a **contact singularity link** *which admits* **infinitely many exotic** *Stein fillings such that the* **fundamental group** *of each filling is G.*

Some **key ingredients** in the proofs :

Luttinger surgery

symplectic sum

Fintushel-Stern knot surgery

SW-invariants.

Lefschetz fibrations on minimal symplectic fillings of lens spaces

Let ξ denote the canonical contact structure on the lens space $L(p, q)$, which is the link of a **cyclic quotient surface singularity**.

The minimal symplectic fillings of $(L(p, q), \xi)$ has been classified by **Lisca' 08** (and **McDuff' 90** for $(L(p, 1), \xi)$).

Theorem [**Bhupal & O.' 16**] *There is an algorithm to describe any minimal symplectic filling of* $(L(p, q), \xi)$ *as an explicit* **genus-zero** *allowable Lefschetz fibration over* D^2. *Moreover, any minimal symplectic filling of* $(L(p, q), \xi)$ *is obtained by a sequence of* **rational blowdowns*** *starting from the minimal resolution of the corresponding cyclic quotient singularity.*

***rational blowdown** is a surgery operation discovered by **Fintushel & Stern' 97**, where a negative definite linear plumbing submanifold is replaced by a rational 4-ball.

The result of **Bhupal & O.' 16** was extended to cover the case of non-cyclic quotient singularities as well by **H. Choi & J. Park' 20.**

Although we have not used it, it also follows by a theorem of **Wendl' 10**, that each minimal symplectic filling of $(L(p, q), \xi)$ is **deformation equivalent** to a **genus-zero** allowable Lefschetz fibration over D^2. This is because $(L(p, q), \xi)$ is known to be planar (**Schönenberger' 05**), i.e., it admits a planar open book that supports ξ.

Lefschetz fibrations and trisections

A **handlebody** is a compact manifold admitting a handle decomposition with a single 0-handle and some 1-handles.

Definition : A **trisection** of a closed 4-manifold X is a decomposition of X into **three** 4D-handlebodies, whose **pairwise** intersections are 3D-handlebodies and whose **triple** intersection is a closed embedded surface.

This is analogous to a **Heegaard splitting** of a closed 3-manifold, which is a decomposition into **two** 3D-handlebodies whose intersection is an embedded surface.

Trisections can be presented by **trisection diagrams** (similar to the Heegaard diagrams).

Theorem (Gay & Kirby' 16)

Every closed oriented 4-manifold admits a trisection.

Gay' 16 also constructed a **trisection** directly from a given **Lefschetz pencil**, *without describing an explicit trisection diagram.*

In a joint work with **Castro**, we obtained an alternate proof of the theorem of **Gay and Kirby' 16** using ***Lefschetz fibrations*** and ***contact geometry*** and we proved the following.

Theorem [**Castro & O.' 19**] Suppose that X is a closed, oriented 4-manifold which admits a Lefschetz fibration over S^2 with a **section of square** -1. Then, an explicit trisection of X can be described by a corresponding trisection diagram, which is determined by the **vanishing cycles** of the Lefschetz fibration.

Baykur and Saeki' 18 obtained an alternate proof of the theorem of **Gay and Kirby' 16**, setting up a correspondence between ***broken*** Lefschetz fibrations and trisections, *using a method which is very different from ours.* They also proved a stronger version of our result.

Example : A trisection diagram for the **Horikawa surface** $H'(1)$, a simply-connected **complex surface of general type** which admits a genus two Lefschetz fibration over S^2. Note that $H'(1)$ is an **exotic copy** of $5\mathbb{C}P^2\#29\overline{\mathbb{C}P^2}$.