Stable solutions to semilinear elliptic equations are smooth up to dimension 9

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Stable solutions to semilinear elliptic equations are smooth up to dimension 9

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Joint work with Alessio Figalli, Xavier Ros-Oton, and Joaquim Serra. Acta Math. 2020

- Semilinear elliptic PDEs: $\quad -\Delta u = f(u)$ in $\Omega \subset \mathbb{R}^n$, bdd domain

Energy: $E_\Omega(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u)$, $F' = f$ $\quad$ 1st variation

2nd variation is $-\Delta - f'(u)$ = linearized operator at u for the equation $-\Delta u = f(u)$

it is nonnegative iff $-\Delta - f'(u) \geq 0$

iff $\displaystyle\int_\Omega f'(u)\xi^2 \leq \int_\Omega |\nabla \xi|^2 \quad \forall \xi \in C_c^\infty(\Omega)$ $\leftarrow$ Def. of stability

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iff $\int_\Omega f'(u)\xi^2 \leq \int_\Omega |\nabla \xi|^2 \quad \forall \xi \in C_c^\infty(\Omega)$ $\leftarrow$ Def. of stability

$\rightarrow$ Competitors $u + \varepsilon\xi$ have all same boundary values as u

$\rightarrow$ Our interest: nonlinearities f superlinear at $+\infty$ & $f \geq 0$

$\Downarrow$

NO absolute minimizer exists

$$E_\Omega(t\varphi) = t^2 \int_\Omega \frac{1}{2}|\nabla \varphi|^2 - \int_\Omega F(t\varphi) \underset{t\to+\infty}{\longrightarrow} -\infty \quad (F(t\varphi) \gg t^2\varphi^2)$$

- The Barenblatt-Gelfand problem 1963 :

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \subset \mathbb{R}^n \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f(0) > 0$, nondecreasing, convex, & superlinear at $+\infty$.

Model nonlinearities: $f(u) = e^u$ (combustion theory)

$f(u) = (1+u)^p$, $p > 1$

- **The Barenblatt-Gelfand problem 1963 :**

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with $f(0) > 0$, nondecreasing, convex, & superlinear at $+\infty$.

- Then, $\exists\, \lambda^* \in (0, +\infty)$ & $0 < \lambda < \lambda^* \Rightarrow \exists\, u_\lambda > 0$ stable classical (L^∞) sol'n

  - $u_\lambda \nearrow u^*$ as $\lambda \nearrow \lambda^*$

    ↳ $u^* \in L^1(\Omega)$ is a distributional stable solution for $\lambda = \lambda^*$

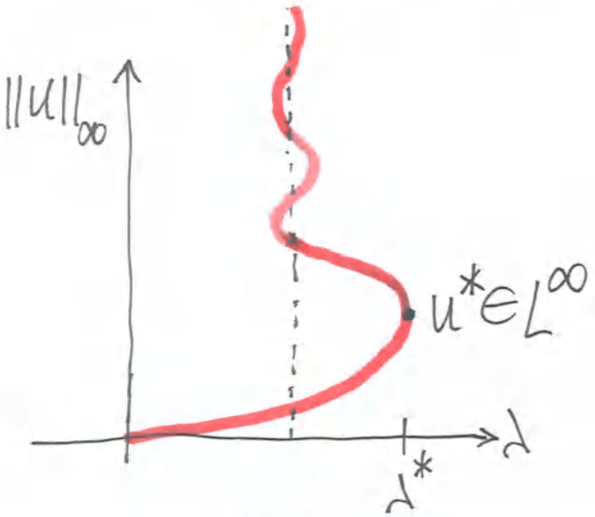
    u^* = the extremal solution of the pb.

- $\nexists$ solutions for $\lambda > \lambda^*$

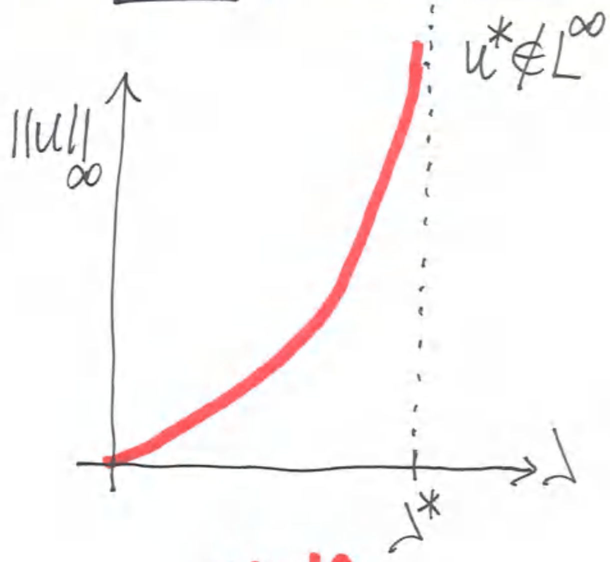
Model nonlinearities: $f(u) = e^u$ (combustion theory)

$f(u) = (1+u)^p$, $p > 1$

• [Joseph-Lundgren '72] $f(u)=e^u$ & $\Omega = B_1$ (RADIAL case):



$3 \leq n \leq 9$



$n \geq 10$

▪ ODE techniques

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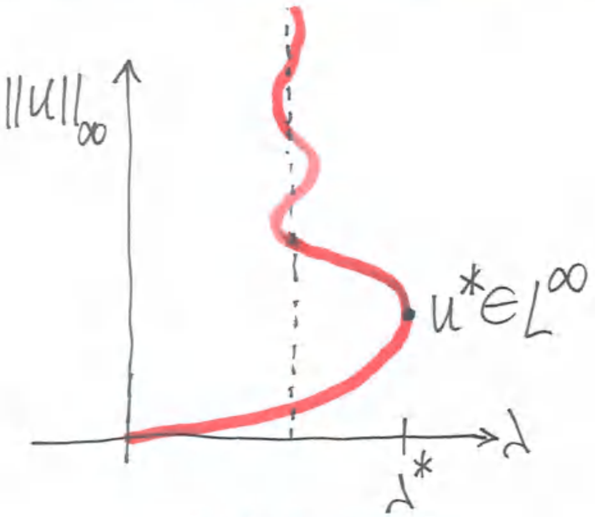

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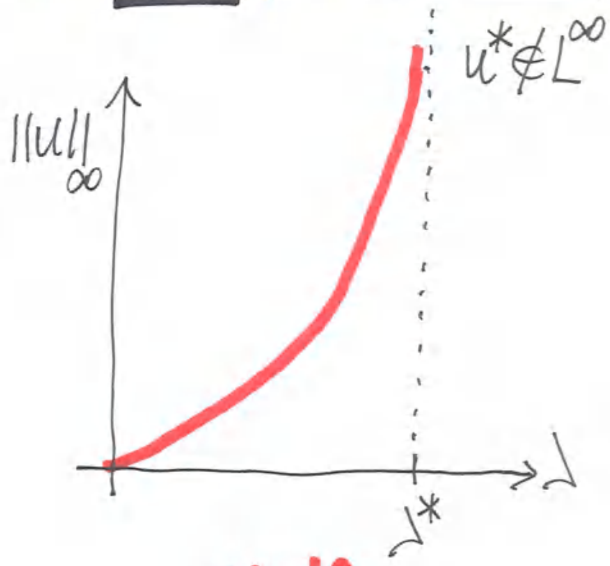

$n \geq 10$

▪ ODE techniques

▪ Explicit singular solution:

$$u(x) = -2\log|x| \in W_0^{1,2}(B_1)$$

solves $-\Delta u = 2(n-2)\,e^u$ in B_1, $n \geq 3$

Linearized operator $= -\Delta - 2(n-2)\dfrac{1}{|x|^2}$

(Hardy's ineq) $\to$ u stable $\Leftrightarrow$ $2(n-2) \leq \dfrac{(n-2)^2}{4}$ $\Leftrightarrow$ $n \geq 10$

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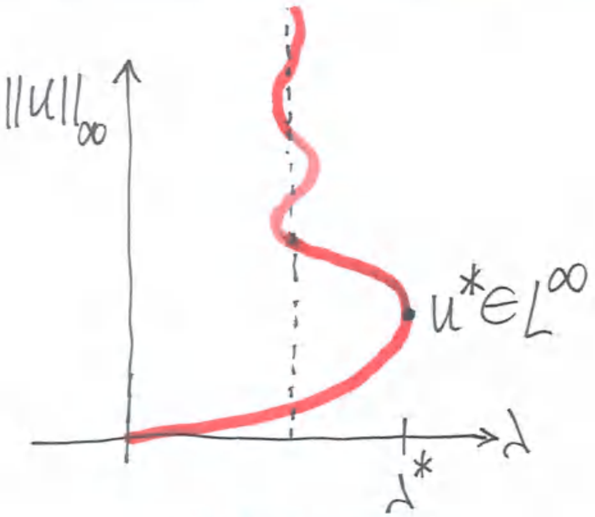

$3 \leq n \leq 9$

$n \geq 10$

- ODE techniques
- Similar for $f(u)=(1+u)^p$

$\downarrow$

explicit solutions

$u(x) = |x|^{-\alpha_p} - 1$

$(\alpha_p > 0)$

- Explicit singular solution:

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- **Questions:** When is $\overline{u^* \in L^\infty(\Omega)}$?
  When are $W_0^{1,2}$ stable solutions bounded?

  - For general solutions, L^∞ estimates exist for f subcritical or critical : $|f(u)| \leq C(1+|u|)^p$, $p \leq \frac{n+2}{n-2}$

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- PDE analogue of "regularity of stable minimal surfaces in $\mathbb{R}^n$":
  - → Not true for $n \geq 8$
  - → True for $n=3$ ([Fischer-Colbrie & Schoen '80] [DoCarmo & Peng '79])
  - → Open pb for $4 \leq n \leq 7$!!

(→ Known for $n \leq 7$ for minimizing minimal surfaces)

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- 1^{st} result $\forall \Omega$: [Crandall-Rabinowitz '75]

  $u^* \in L^\infty(\Omega)$ if $n \leq 9$ and $f(u) \sim e^u$ or $f(u) \sim (1+u)^p$

- [Brezis-Vázquez '97] Is it always $u^* \in W_0^{1,2}(\Omega)$?
- [Brezis '03] Is there something "sacred" about dim 10 ?
  Is it possible to construct a singular stable sol'n for $n \leq 9$, in some domain & for some f ?

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- [Villegas '13] $u^* \in L^\infty(\Omega)$ if $n \leq 4$; $u^* \in W_0^{1,2}(\Omega)$ if $n \leq 6$

- [Cabré & Ros-Oton '13] L^∞ if $n \leq 7$ & Ω of double revolution
- [Cabré-Sanchón-Spruck '16] L^∞ if $n \leq 5$ & $f'/f^{1+\varepsilon} \leq C(\varepsilon)$ $\forall \varepsilon > 0$

■ [Cabré, Figalli, Ros-Oton, Serra '19]

Thm 1 $u \in C^2(B_1)$ stable sol'n of $-\Delta u = f(u)$ in B_1 & $f \geq 0$ $\Rightarrow$

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/2})} \leq C(n)\,\|u\|_{L^1(B_1)} \qquad (\gamma = \gamma(n) > 0)$$

& if $n \leq 9$ then $\|u\|_{C^\alpha(\overline{B}_{1/2})} \leq C(n)\,\|u\|_{L^1(B_1)}$ $(\alpha = \alpha(n) > 0)$.

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Thm 2 Ω bdd C^3 domain, $f \geq 0$, $f' \geq 0$, $f'' \geq 0$.
$u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ stable sol'n of $\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$ $\Bigg\}$ $\Rightarrow$

$$\|\nabla u\|_{L^{2+\gamma}(\Omega)} \leq C(\Omega) \|u\|_{L^1(\Omega)} \qquad (\gamma = \gamma(n) > 0)$$

& if $n \leq 9$ then $\|u\|_{C^\alpha(\overline{\Omega})} \leq C(\Omega) \|u\|_{L^1(\Omega)}$ $(\alpha = \alpha(n) > 0)$.

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Corol 2 Ω bdd C^3 domain $\Longrightarrow$ $\begin{cases} u^* \in W_0^{1,2+\gamma}(\Omega) \quad (\gamma = \gamma(n) > 0) \\ \text{if } n \le 9,\ u^* \in L^\infty(\Omega). \end{cases}$

Thm 3 Sharp Morrey $M^{p,\beta}(\Omega)$ estimates for stable sol's when $n \geq 10$.

RELATED WORK:

- **p-Laplacian** $\quad -\Delta_p u = f(u)$, $\quad 1<p<2$
  - [Cabré-Miraglio-Sanchón '20] Optimal result for $p>2$: regularity if $n < p + \frac{4p}{p-1}$.
  - Optimal result is open for $1<p<2$.

- **Fractional Laplacian** $\quad (-\Delta)^s u = f(u)$, $\quad 0<s<1$
  - Optimal dimensions: open even in the radial case

    involved relation on Γ-function: only known for $f(u)=e^u$ in convex symmetric domains [Ros-Oton '14]

- PROOFS

$$\Delta u + f(u) = 0 \qquad \text{(EQUATION)}$$

$$\downarrow$$

$$\Delta + f'(u) \qquad \text{(LINEARIZED OPERATOR} \leq 0\text{)}$$

$$\downarrow$$

$$\int_\Omega f'(u)\,\xi^2 \leq \int_\Omega |\nabla \xi|^2 \quad \forall \xi \in C_c^1(\Omega) \qquad \text{(STABILITY)}$$

$$\Big\downarrow\ \xi = c\cdot\eta \qquad \text{with } \eta_{|\partial\Omega} \equiv 0.$$

$$\int_\Omega c\,\underline{(\Delta c + f'(u)c)}\,\eta^2 \leq \int_\Omega c^2 |\nabla \eta|^2.$$

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$$\int_\Omega f'(u)\,\xi^2 \leq \int_\Omega |\nabla \xi|^2 \qquad \forall \xi \in C^1_c(\Omega)$$

$\xi = c\cdot\eta$ with $\eta_{|\partial\Omega} \equiv 0$

$$\int_\Omega c\,\underline{(\Delta c + f'(u)c)}\,\eta^2 \leq \int_\Omega c^2 |\nabla \eta|^2 .$$

[Cabré-Capella '05]
motivated by
Simons' lemma on
minimal cones

$c = \|$second fund. form$\|$

- **Proofs** $\xi = c\cdot\eta \qquad \Longrightarrow \int_\Omega c(\Delta c + f'(u)c)\,\eta^2 \leq \int_\Omega c^2|\nabla\eta|^2. \qquad (\eta_{|\partial\Omega} \equiv 0)$

- [Crandall-Rabinowitz] & [Nedev] : $\underline{\xi = h(u)}$

- [Cabré-Capella] $(\Omega = B_1)$: $\xi = \underbrace{ru_r}_{\substack{= \\ c \\ = \\ x\cdot\nabla u}} \cdot \underbrace{r^{-a}\varphi}_{\substack{= \\ \eta \\ \\ |x|^{-a}\cdot\varphi}}$, φ cut-off near ∂B_1

- [Cabré '10] $(n \leq 4)$: $\xi = \underbrace{|\nabla u|}_{\substack{= \\ c}} \cdot \underbrace{g(u)}_{\substack{= \\ \eta}}$

- **Proofs** $\xi = c\cdot\eta$ $\quad\Rightarrow \displaystyle\int_\Omega c(\Delta c + f'(u)c)\,\eta^2 \leq \int_\Omega c^2|\nabla\eta|^2.$ $\quad(\eta_{|\partial\Omega} \equiv 0)$

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$\qquad |x|^{-a}\cdot\varphi$

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For our interior result $(n\leq 9)$ we will use both
$c = x\cdot\nabla u$
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$$\underline{(\Delta + f'(u))(x\cdot\nabla u)} = 2\Delta u$$

&

$$\underline{(\Delta + f'(u))|\nabla u|} = \frac{1}{|\nabla u|}\left\{\sum_{i,j} u_{ij}^2 - \sum_i\left(\sum_j u_{ij}\frac{u_j}{|\nabla u|}\right)^2\right\}$$

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Curvature of level sets ⊕ Michael-Simon Sobolev ineq.

Using $\xi = c\eta = (x\cdot\nabla u)\,\eta(x) \rightsquigarrow \int_\Omega (x\cdot\nabla u)\, 2\Delta u\, \eta^2 \rightsquigarrow$ Pohozaev trick

Lemma 1 $\forall n\ \forall f\ \forall u$ stable sol'n $\forall \eta \in C^1_c(B_1) \Longrightarrow$

$$\int_{B_1} \{(n-2)\eta + 2x\cdot\nabla\eta\}\,\eta\,|\nabla u|^2 - 2(x\cdot\nabla u)\,\nabla u\cdot\nabla(\eta^2) - |x\cdot\nabla u|^2|\nabla\eta|^2 \leq 0.$$

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(0) (1) (2)

$$\boxed{\xi = (x\cdot\nabla u)\,\underbrace{|x|^{\frac{2-n}{2}}\,\varphi(x)}_{=\,\eta(x)}}$$

so that

(0) $\{\cdots\} \ge 0$

Using $\xi = c\eta = (x\cdot\nabla u)\,\eta(x) \rightsquigarrow \int_\Omega (x\cdot\nabla u)\, 2\Delta u\, \eta^2$ $\rightsquigarrow$ Pohozaev trick

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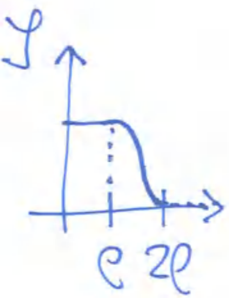
$$\int_{B_1} \underbrace{\{(n-2)\eta + 2x\cdot\nabla\eta\}}_{(0)}\,\eta\,|\nabla u|^2 - 2(x\cdot\nabla u)\,\nabla u\cdot\nabla(\eta^2) \quad (1)$$

$$(2)\quad -|x\cdot\nabla u|^2\,|\nabla\eta|^2 \le 0.$$

$$\boxed{\xi = (x\cdot\nabla u)\,\underbrace{|x|^{\frac{2-n}{2}}\varphi(x)}_{=\,\eta(x)}}$$

so that $\{\cdots\} \ge 0$ (0)



(1) & (2) $\longrightarrow$ $2(n-2) - \frac{(n-2)^2}{4} = \frac{1}{4}\{8(n-2) - (n-2)^2\}$

$$= \frac{1}{4}(n-2)(10-n)$$

$$\frac{1}{4}(n-2)(10-n)\int_{B_\rho} |x|^{2-n} u_r^2 \le C \int_{B_{2\rho}\setminus B_\rho} |x|^{2-n}|\nabla u|^2$$

NOTE: $\displaystyle\int_{B_{1/2}} |x-y|^{2-n} \left|\nabla u(x)\cdot\frac{x-y}{|x-y|}\right|^2 dx \leqslant C \qquad \forall y \in B_{1/2}$

$\hookrightarrow$ $u \in BMO$ if $n \leq 9$

easy

WE HAVE:

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$\xrightarrow[\text{easy}]{}$ $u \in BMO$ if $n \leq 9$

WE HAVE:

$$\int_{B_\rho} |x|^{2-n} u_r^2 \leq C \int_{B_{2\rho}\setminus B_\rho} |x|^{2-n} |\nabla u|^2$$

If we had $\displaystyle\int_{B_{2\rho}\setminus B_\rho} |x|^{2-n}|\nabla u|^2 \leq C' \int_{B_{2\rho}\setminus B_\rho} |x|^{2-n} u_r^2$, then $\rightarrow$

$$\rightarrow \int_{B_\rho} |x|^{2-n} u_r^2 \leq C'' \int_{B_{2\rho}\setminus B_\rho} |x|^{2-n} u_r^2$$

$$\rightarrow \int_{B_\rho} |x|^{2-n} u_r^2 \leq \frac{C''}{1+C''} \overbrace{\int_{B_{2\rho}} |x|^{2-n} u_r^2}^{\text{adimensional quantity}} \Longrightarrow \text{Algebraic decay \& Hölder continuity of } u$$

We would like $\displaystyle\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2 \leq C(n) \int_{B_{1/2}\setminus B_{1/4}} u_r^2 .$ **(*)**

May it be true **?**

We would like
$$\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2 \leq C(n) \int_{B_{1/2}\setminus B_{1/4}} u_r^2 . \qquad (*)$$

May it be true?

If false, in the extreme case we would have

$$\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2 = 1 \quad \& \quad \int_{B_{1/2}\setminus B_{1/4}} u_r^2 = 0$$

CONTRADICTION

$\rightarrow$ u is 0-homogeneous

$\Downarrow$ $-\Delta u = f(u) \geq 0$

$u = ctt \Leftarrow$ u is a superharmonic f'cn on the sphere S^{n-1}

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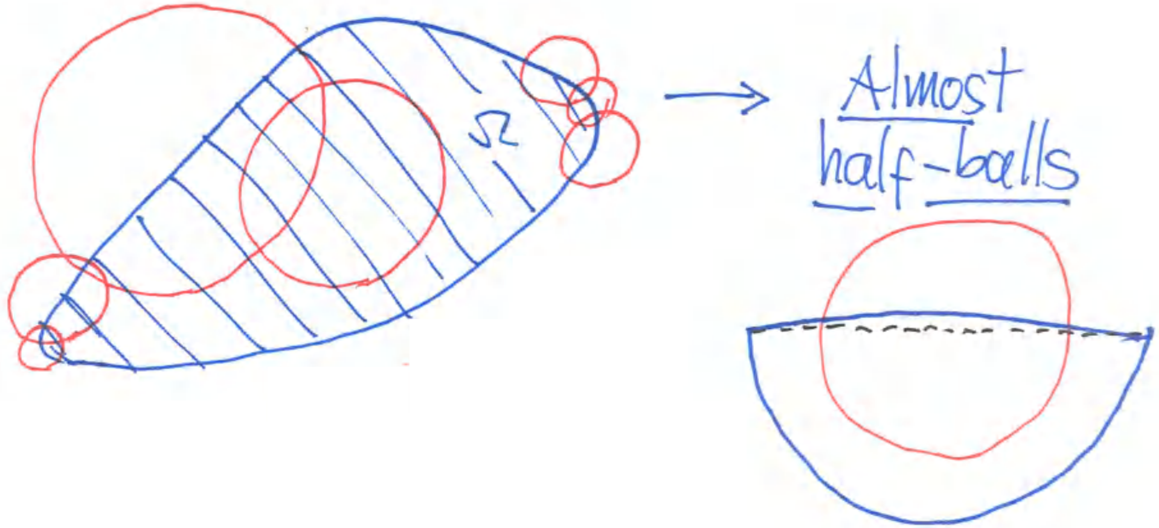
$u = ctt \Leftarrow$

CONTRADICTION

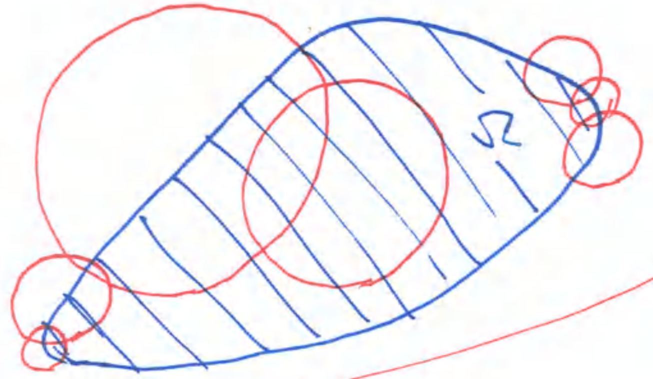
$\to$ We prove $(*)$ (under a doubling assumption that suffices) by COMPACTNESS using the higher integrability estimate

$$\boxed{c = |\nabla u|} \Longrightarrow \|\nabla u\|_{L^{2+\gamma}} \leq C(n) \|\nabla u\|_{L^2}$$

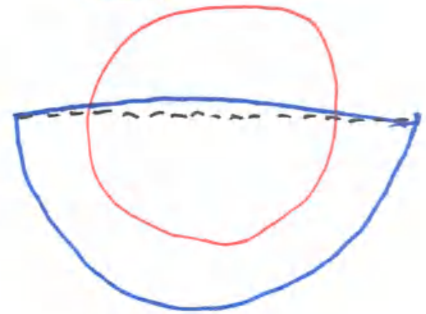
- **Boundary regularity**



• Boundary regularity

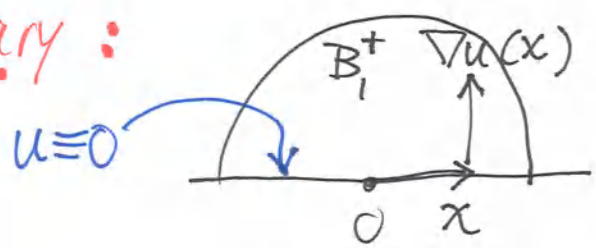


Almost half-balls



Simplest case:

Half-balls ; flat boundary :



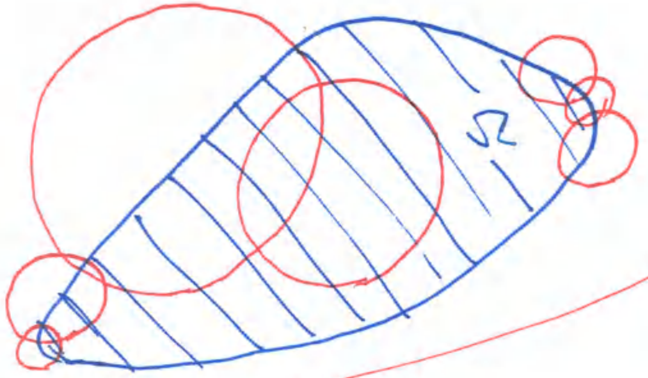
$\downarrow$

$$\xi(x) = (x \cdot \nabla u)\,|x|^{\frac{2-n}{2}}\,\varphi(x)$$ vanishes on the flat bdry ☺

$$\Downarrow$$

$$(n-2)(10-n)\int_{B_\rho^+} |x|^{2-n} u_r^2 \leq C \int_{B_{2\rho}^+ \setminus B_\rho^+} |x|^{2-n} |\nabla u|^2$$

- **Boundary regularity**



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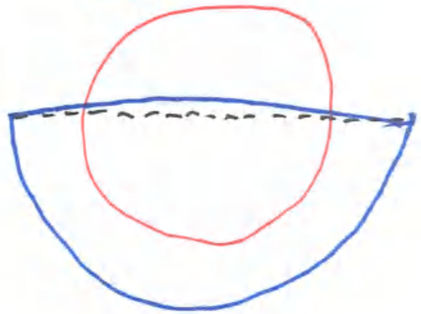

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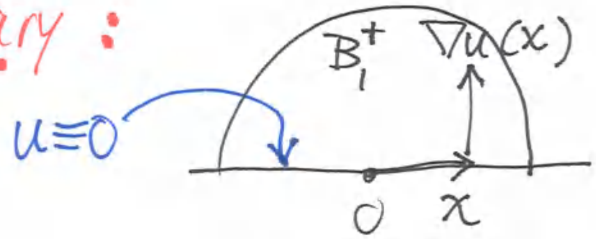

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$\xi(x) = (x\cdot\nabla u)\,|x|^{\frac{2-n}{2}}\,\varphi(x)$ vanishes on the flat bdry ☺

$\Downarrow$

$$(n-2)(10-n)\int_{B_\rho^+} |x|^{2-n} u_r^2 \leq C \int_{B_{2\rho}^+\setminus B_\rho^+} |x|^{2-n}|\nabla u|^2$$

We ask ¿ $\exists u$, $u=0$ on $\{x_n=0\}$, $\Delta u \leq 0$ in $\{x_n>0\}$, **Yes** ☹

$$\int_{B_2^+\setminus B_1^+} |\nabla u|^2 = 1 \quad \& \quad \int_{B_2^+\setminus B_1^+} u_r^2 = 0 \ ?$$

$u(r,\theta) = \sin\theta$

key remark: u cannot solve $-\Delta u = f(u)$ if $u = u(\theta)$

0 homogeneous

-2 homogeneous

Question: Can one pass to the limit the condition $-\Delta u = f(u)$?

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Thm 4 Let u_k be stable sol'ns of $-\Delta u_k = f_k(u_k)$ in $U \subset \mathbb{R}^n$ open, with $f_k' \geq 0$, $f_k'' \geq 0$; $u_k \in W^{1,2}_{loc}(U)$

$$u_k \xrightarrow{\text{in } L^1_{loc}(U)} u.$$

Then $u \in W^{1,2}_{loc}(U)$ is a stable solution of $-\Delta u = f(u)$ in U for some f nondecreasing and convex, $f : (-\infty, M) \to \mathbb{R}$.

Thanks for your attention