Positive periodic solutions for nonlinear delay dynamic equations on time scales

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In this work, we use fixed point theorem to study the existence of positive periodic solutions for delay dynamic equation on time scales. Transforming the equation to an integral equation enables to show the existence of positive periodic solutions by appealing to Krasnoselskii's fixed point theorem. The obtained integral equation is the sum of two mappings; one is a contraction and the other is compact.

1. Introduction

Let $\mathbb{T}$ be a periodic time scale with $0 \in \mathbb{T}$. The aim of this work is to extend the results obtained in [10] to the first-order neutral delay dynamic equation on a time scale. More precisely, we consider the equation

$$\left(r(t)\left[x(t) - P(t)\int_c^d x(t-\tau(t,\xi))\,\Delta\xi\right]\right)^{\Delta}$$
$$(1.1)\qquad = -Q(t)\,x^{\sigma}(t) + \int_c^d f(t, x(t-\tau(t,\xi)))\,\Delta\xi,$$

where $r \in C^1_{rd}(\mathbb{T},\mathbb{R})$, $r(t) > 0$, $\tau \in C_{rd}(\mathbb{T}\times[c,d]_{\mathbb{T}},(0,\infty)_{\mathbb{T}})$, $Q \in C_{rd}(\mathbb{T},(0,\infty))$, $P \in C_{rd}(\mathbb{T},\mathbb{R})$, $f \in C_{rd}(\mathbb{T}\times\mathbb{R},\mathbb{R})$, $d > c \geq 0$, and r, P, Q are ω-periodic functions, f is ω-periodic with respect to first variable, and τ is ω-periodic with respect to t.

2. The inversion and the fixed point theorem

Theorem 3.1 ([15]). *Let $\mathbb{M}$ be a bounded closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Assume $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) $x, y \in \mathbb{M}$ imply $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$,

(ii) $\mathcal{A}$ is completely continuous,

(iii) $\mathcal{B}$ is a contraction mapping.

Then, there exists $z \in \mathbb{M}$ with $\mathcal{A}z + \mathcal{B}z = z$.

Lemma 3.1 ([14]). *Let $x \in \Phi$. Then $\|x^{\sigma}\|$ exists and $\|x^{\sigma}\| = \|x\|$.*

Theorem 3.2. *Assume $0 \leq P(t)(d-c) \leq p_1 < 1$ and there exist positive constants m and M with $0 < m < M$ such that*

$$(3.1)\qquad \frac{r_1 m}{(d-c)} \leq r^{\sigma}(t)\left[\frac{f(t,x)}{Q(t)} - P^{\sigma}(t)\,x^{\sigma}(t)\right] \leq \frac{r_0(1-p_1)M}{(d-c)}, \quad \forall (t,x) \in [0,\omega]_{\mathbb{T}} \times [m,M].$$

Then, (1.1) has a positive ω-periodic solution $x \in [m,M]$.

$$\left(r(t)\left[x(t) - P(t)\int_c^d x(t-\tau(t,\xi))\,\Delta\xi\right]\right)^{\Delta}$$
$$= -Q(t)\,x^{\sigma}(t) + \int_c^d f(t,x(t-\tau(t,\xi)))\,\Delta\xi$$
$$= -Q(t)\left[x^{\sigma}(t) - P^{\sigma}(t)\int_c^d x^{\sigma}(t-\tau(t,\xi))\,\Delta\xi\right]$$
$$+ \int_c^d f(t,x(t-\tau(t,\xi)))\,\Delta\xi - Q(t)\,P^{\sigma}(t)\int_c^d x^{\sigma}(t-\tau(t,\xi))\,\Delta\xi$$
$$- A(t)\left(r(t)\left[x(t) - P(t)\int_c^d x(t-\tau(t,\xi))\,\Delta\xi\right]\right)^{\sigma}$$
$$(3.2)\qquad + \int_c^d f(t,x(t-\tau(t,\xi))) - Q(t)\,P^{\sigma}(t)\,x^{\sigma}(t-\tau(t,\xi))\,\Delta\xi,$$

where

$$A(t) := \frac{Q(t)}{r^{\sigma}(t)}.$$

Multiplying through (3.2) by $e_A(t,0)$ gives

$$\left[r(t)\left(x(t) - P(t)\int_c^d x(t-\tau(t,\xi))\,\Delta\xi\right)e_A(t,0)\right]^{\Delta}$$
$$(3.3)\qquad = \left[\int_c^d f(t,x(t-\tau(t,\xi))) - Q(t)\,P^{\sigma}(t)\,x^{\sigma}(t-\tau(t,\xi))\,\Delta\xi\right]e_A(t,0)$$

Integrating (3.3) from t to $t+\omega$ gives

$$x(t) = \frac{1}{r(t)}\int_t^{t+\omega} G(t,s)\int_c^d f(s,x(s-\tau(s,\xi))) - Q(s)\,P^{\sigma}(s)\,x^{\sigma}(s-\tau(s,\xi))\,\Delta\xi\Delta s$$
$$+ P(t)\int_c^d x(t-\tau(t,\xi))\,\Delta\xi,$$

where

$$G(t,s) = \frac{e_A(s,t)}{e_A(\omega,0)-1}.$$

Let $\mathbb{M} = \{x \in \Phi : m < x(t) < M,\ t \in [0,\omega]_{\mathbb{T}}\}$. We observe that $\mathbb{M}$ is a bounded, closed, and convex subset of Φ. Define two mappings $\mathcal{A}, \mathcal{B} : \mathbb{M} \to \Phi$ as follows

$$(3.4)\qquad (\mathcal{A}x)(t) = \frac{1}{r(t)}\int_t^{t+\omega} G(t,s)\int_c^d f(s,x(s-\tau(s,\xi))) - Q(s)\,P^{\sigma}(s)\,x^{\sigma}(s-\tau(s,\xi))\,\Delta\xi\Delta s,$$

$$(3.5)\qquad (\mathcal{B}x)(t) = P(t)\int_c^d x(t-\tau(t,\xi))\,\Delta\xi.$$

Theorem 3.3. *Assume $-1 < p_0 \leq P(t)(d-c) \leq 0$ and there exist positive constants m and M with $0 < m < M$ such that*

$$(3.6)\qquad \frac{r_1(m-p_0M)}{d-c} \leq r^{\sigma}(t)\left[\frac{f(t,x)}{Q(t)} - P^{\sigma}(t)\,x^{\sigma}(t)\right] \leq \frac{r_0 M}{d-c}, \quad \forall (t,x) \in [0,\omega]_{\mathbb{T}} \times [m,M].$$

Then, (1.1) has a positive ω-periodic solution $x \in [m,M]$.

Corollary 3.1. *Assume $r(t) = 1$, $0 \leq P(t)(d-c) \leq p_1 < 1$ and there exist positive constants m and M with $0 < m < M$ such that*

$$\frac{m}{(d-c)} \leq \left[\frac{f(t,x)}{Q(t)} - P^{\sigma}(t)\,x^{\sigma}(t)\right] \leq \frac{(1-p_1)M}{(d-c)}, \quad \forall (t,x) \in [0,\omega]_{\mathbb{T}} \times [m,M].$$

Then, (1.1) has a positive ω-periodic solution $x \in [m,M]$.

Corollary 3.2. *Assume $r(t) = 1$, $-1 < p_0 \leq P(t)(d-c) \leq 0$ and there exist positive constants m and M with $0 < m < M$ such that*

$$\frac{(m-p_0M)}{d-c} \leq \left[\frac{f(t,x)}{Q(t)} - P^{\sigma}(t)\,x^{\sigma}(t)\right] \leq \frac{M}{d-c}, \quad \forall (t,x) \in [0,\omega]_{\mathbb{T}} \times [m,M].$$

Then, (1.1) has a positive ω-periodic solution $x \in [m,M]$.

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