## Infinitely many entire solutions to the curl-curl problem with critical exponent

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Joint work with Michał Gaczkowski and Jarosław Mederski

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#### Physical derivation

Maxwell's equations in absence of currents and magnetization:

$$\begin{cases} \nabla \times (\mu^{-1}\mathcal{B}) = \frac{\partial(\varepsilon \mathcal{E} + \mathcal{P})}{\partial t} & (\mathsf{Ampère's law}) \\ \nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t} & (\mathsf{Faraday's law}) \end{cases}$$

where  $\mathcal{E}$  is the electric field,  $\mathcal{B}$  is the magnetic induction,  $\mathcal{P}$  is the polarization,  $\varepsilon$  is the permittivity and  $\mu$  is the permeability. Taking  $\frac{\partial}{\partial t}$  in Ampère's law and  $\varepsilon$  and  $\mu$  constant in time,

$$-\nabla \times \left(\mu^{-1}\nabla \times \mathcal{E}\right) = \nabla \times \left(\mu^{-1}\frac{\partial \mathcal{B}}{\partial t}\right) = \varepsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} + \frac{\partial^2 \mathcal{P}}{\partial t^2}.$$

With monochromatic waves, i.e.,  $\mathcal{E}(x, t) = E(x) \cos(\omega t)$  and  $\mathcal{P}(x, t) = P(x) \cos(\omega t)$ ,

$$\nabla \times (\mu^{-1} \nabla \times E) - \varepsilon \omega^2 E = \omega^2 P.$$

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Find infinitely many solutions to

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•  $\nabla \times \nabla \phi = 0$  for every  $\phi \in \mathcal{C}^{\infty}_{c}(\mathbb{R}^{3})$ •  $\nabla \times \nabla \times \Phi = -\Delta \Phi + \nabla (\nabla \cdot \Phi)$  for every  $\Phi \in \mathcal{C}^{2}(\mathbb{R}^{3}, \mathbb{R}^{3})$ Solutions to (1) in 1-to-1 correspondence with critical points of

$$J\colon \mathbf{U}\in \mathcal{D}^{1,2}(\mathbb{R}^3,\mathbb{R}^3)\mapsto \int_{\mathbb{R}^3}\frac{1}{2}\left|\nabla\times\mathbf{U}\right|^2-\frac{1}{6}\left|\mathbf{U}\right|^6\,\mathrm{d} x\in\mathbb{R}.$$

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#### From abla imes abla imes to $-\Delta$

Let  $\mathcal{SO} := \mathcal{SO}(2) \times \{1\}$  and define  $\mathcal{D}_{\mathcal{SO}} := \{ \ \mathbf{U} \in \mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R}^3) \ | \ \mathbf{U}(g \cdot) = g \mathbf{U} \ \forall g \in \mathcal{SO} \}.$ 

Theorem (A. Azzollini, V. Benci, T. D'Aprile, D. Fortunato; 2006) For all  $\mathbf{U} \in \mathcal{D}_{SO}$  there exist  $\mathbf{U}_{\rho}, \mathbf{U}_{\tau}, \mathbf{U}_{\zeta} \in \mathcal{D}_{SO}$  such that for a.e.  $x \in \mathbb{R}^3$ •  $\mathbf{U}_{\rho}(x)$  is the orthogonal projection onto span $\{(x_1, x_2, 0)\},$ •  $\mathbf{U}_{\tau}(x)$  is the orthogonal projection onto span $\{(-x_2, x_1, 0)\},$ •  $\mathbf{U}_{\zeta}(x)$  is the orthogonal projection onto span $\{(0, 0, 1)\},$ U =  $\mathbf{U}_{\rho} + \mathbf{U}_{\tau} + \mathbf{U}_{\zeta}, \nabla \cdot \mathbf{U}_{\tau} = 0$ , and  $\nabla \mathbf{U}_{\rho}(x), \nabla \mathbf{U}_{\tau}(x), \nabla \mathbf{U}_{\zeta}(x)$  are pairwise orthogonal in  $\mathbb{R}^{3 \times 3} \simeq \mathbb{R}^9$  for almost every  $x \in \mathbb{R}^3$ .

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Theorem (A. Azzollini, V. Benci, T. D'Aprile, D. Fortunato; 2006) For all  $\mathbf{U} \in \mathcal{D}_{S\mathcal{O}}$  there exist  $\mathbf{U}_{\rho}, \mathbf{U}_{\tau}, \mathbf{U}_{\zeta} \in \mathcal{D}_{S\mathcal{O}}$  such that for a.e.  $x \in \mathbb{R}^3$ •  $\mathbf{U}_{\rho}(x)$  is the orthogonal projection onto span $\{(x_1, x_2, 0)\}$ , •  $\mathbf{U}_{\tau}(x)$  is the orthogonal projection onto span $\{(-x_2, x_1, 0)\}$ , •  $\mathbf{U}_{\zeta}(x)$  is the orthogonal projection onto span $\{(0, 0, 1)\}$ , U =  $\mathbf{U}_{\rho} + \mathbf{U}_{\tau} + \mathbf{U}_{\zeta}, \nabla \cdot \mathbf{U}_{\tau} = 0$ , and  $\nabla \mathbf{U}_{\rho}(x), \nabla \mathbf{U}_{\tau}(x), \nabla \mathbf{U}_{\zeta}(x)$  are pairwise orthogonal in  $\mathbb{R}^{3\times3} \simeq \mathbb{R}^9$  for almost every  $x \in \mathbb{R}^3$ .

From  $\nabla \times \nabla \times$  to  $-\Delta$ Consider  $S: \mathcal{D}_{S\mathcal{O}} \to \mathcal{D}_{S\mathcal{O}}$  defined as

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Moreover, for every  $U \in \mathcal{D}_S$ 

$$J(\mathbf{U}) = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \mathbf{U}|^2 - \frac{1}{6} |\mathbf{U}|^6 \, \mathrm{d}x.$$

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## The (scalar) Yamabe Problem

Consider

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Solutions in 1-to-1 correspondence via the isometric isomorphism

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where  $\pi : \mathbb{S}^3 \to \mathbb{R}^3 \cup \{\infty\}$  is the stereographic projection and  $\varphi(x) = \sqrt{\frac{2}{1+|x|^2}}.$ 

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$$\mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{R}^2 imes \mathbb{R}^2,$$
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For  $g = (g_1, g_2) \in SO(2)^2$ ,  $v \in H^1(\mathbb{S}^3)$ , and  $\xi = (\xi_1, \xi_2) \in \mathbb{S}^3$  define  $\gamma_g v(\xi) := v(g_1\xi_1, g_2\xi_2)$  and

 $H_{\mathcal{SO}(2\times 2)} := \left\{ \left. v \in H^1(\mathbb{S}^3) \right| \gamma_g v = v \, \forall g \in \mathcal{SO}(2)^2 \right\}.$ 

Theorem (W. Ding, 1986)  $H_{SO(2\times 2)} \hookrightarrow L^6(\mathbb{S}^3).$ 

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$$S^{3} \subset \mathbb{R}^{4} = \mathbb{R}^{2} \times \mathbb{R}^{2},$$

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For  $g = (g_{1}, g_{2}) \in SO(2)^{2}, v \in H^{1}(\mathbb{S}^{3}), \text{ and } \xi = (\xi_{1}, \xi_{2}) \in \mathbb{S}^{3}$  define  $\gamma_{g} v(\xi) := v(g_{1}\xi_{1}, g_{2}\xi_{2})$  and

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What is  $\gamma_g$  for  $u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ ? If  $v \in H^1(\mathbb{S}^3)$  is defined via (2), then

$$v(\xi) = v(g\xi) \Leftrightarrow \frac{u(x)}{\varphi(x)} = \frac{u(\pi(g\pi^{-1}(x)))}{\varphi(\pi(g\pi^{-1}(x)))}, \quad g = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \in \mathcal{SO}(2)^2.$$

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#### Lemma

- $\mathcal{D}_{\mathcal{SO}(2\times 2)} \hookrightarrow \mathcal{L}^6(\mathbb{R}^3, \mathbb{R}^3).$
- $\mathcal{D}_{\mathcal{SO}(2\times 2)} \subset \mathcal{D}_{\mathcal{SO}}$ .
- If  $\mathbf{U} \in \mathcal{D}_{\mathcal{SO}(2\times 2)}$ , then  $\mathbf{U}_{\rho}, \mathbf{U}_{\tau}, \mathbf{U}_{\zeta} \in \mathcal{D}_{\mathcal{SO}(2\times 2)}$ .
- dim  $Y = \infty$ , where  $Y := \{ \mathbf{U} \in \mathcal{D}_{SO(2 \times 2)} \mid S\mathbf{U} = \mathbf{U} \} = \{ \mathbf{U} \in \mathcal{D}_{SO(2 \times 2)} \mid \mathbf{U} = \mathbf{U}_{\tau} \}.$

#### Theorem (P. H. Rabinowitz, 1986)

*E* Banach Space, dim $E = \infty$ ,  $I \in C^1(E)$  even, I(0) = 0 and satisfying (*PS*) at every positive level,  $\inf_{S_r} I > 0$  for some r > 0. Assume for every finite-dimensional subspace  $F \subset E$  there exists R(F) > 0 such that  $\sup_{F \setminus B_{R(F)}} I \leq 0$ . Then there exists  $x_n \in E$  such that  $I'(x_n) = 0$  and  $I(x_n) \to \infty$ .

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# Thank you for your attention!

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abla imes U = |U|^4 U$  in  $\mathbb{R}^3$ 

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