Infinitely many entire solutions to the curl-curl problem with critical exponent

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Joint work with Michał Gaczkowski and Jarosław Mederski

Physical derivation

Maxwell's equations in absence of currents and magnetization:

$$\begin{cases} \nabla \times (\mu^{-1}\mathcal{B}) = \frac{\partial(\varepsilon\mathcal{E}+\mathcal{P})}{\partial t} & \text{(Ampère's law)} \\ \nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t} & \text{(Faraday's law)} \end{cases}$$

where $\mathcal{E}$ is the electric field, $\mathcal{B}$ is the magnetic induction, $\mathcal{P}$ is the polarization, ε is the permittivity and μ is the permeability.
Taking $\frac{\partial}{\partial t}$ in Ampère's law and ε and μ constant in time,

$$-\nabla \times (\mu^{-1}\nabla \times \mathcal{E}) = \nabla \times \left(\mu^{-1}\frac{\partial \mathcal{B}}{\partial t}\right) = \varepsilon\frac{\partial^2 \mathcal{E}}{\partial t^2} + \frac{\partial^2 \mathcal{P}}{\partial t^2}.$$

With monochromatic waves, i.e., $\mathcal{E}(x,t) = E(x)\cos(\omega t)$ and $\mathcal{P}(x,t) = P(x)\cos(\omega t)$,

$$\nabla \times (\mu^{-1}\nabla \times E) - \varepsilon\omega^2 E = \omega^2 P.$$

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The curl-curl problem

Find infinitely many solutions to

$$\nabla \times \nabla \times \mathbf{U} = |\mathbf{U}|^4 \mathbf{U}, \quad \mathbf{U} \in \mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R}^3). \tag{1}$$

- $\nabla \times \nabla \phi = 0$ for every $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$
- $\nabla \times \nabla \times \Phi = -\Delta \Phi + \nabla(\nabla \cdot \Phi)$ for every $\Phi \in \mathcal{C}^2(\mathbb{R}^3, \mathbb{R}^3)$

Solutions to (1) in 1-to-1 correspondence with critical points of

$$J \colon \mathbf{U} \in \mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \times \mathbf{U}|^2 - \frac{1}{6} |\mathbf{U}|^6 \, dx \in \mathbb{R}.$$

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From $\nabla \times \nabla \times$ to $-\Delta$

Let $\mathcal{SO} := \mathcal{SO}(2) \times \{1\}$ and define

$$\mathcal{D}_{\mathcal{SO}} := \left\{ \mathbf{U} \in \mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R}^3) \mid \mathbf{U}(g\cdot) = g\mathbf{U} \, \forall g \in \mathcal{SO} \right\}.$$

Theorem (A. Azzollini, V. Benci, T. D'Aprile, D. Fortunato; 2006)

For all $\mathbf{U} \in \mathcal{D}_{\mathcal{SO}}$ *there exist* $\mathbf{U}_\rho, \mathbf{U}_\tau, \mathbf{U}_\zeta \in \mathcal{D}_{\mathcal{SO}}$ *such that for a.e.* $x \in \mathbb{R}^3$

- $\mathbf{U}_\rho(x)$ *is the orthogonal projection onto* $\mathrm{span}\{(x_1, x_2, 0)\}$,
- $\mathbf{U}_\tau(x)$ *is the orthogonal projection onto* $\mathrm{span}\{(-x_2, x_1, 0)\}$,
- $\mathbf{U}_\zeta(x)$ *is the orthogonal projection onto* $\mathrm{span}\{(0, 0, 1)\}$,

$\mathbf{U} = \mathbf{U}_\rho + \mathbf{U}_\tau + \mathbf{U}_\zeta$, $\nabla \cdot \mathbf{U}_\tau = 0$, *and* $\nabla \mathbf{U}_\rho(x)$, $\nabla \mathbf{U}_\tau(x)$, $\nabla \mathbf{U}_\zeta(x)$ *are pairwise orthogonal in* $\mathbb{R}^{3\times 3} \simeq \mathbb{R}^9$ *for almost every* $x \in \mathbb{R}^3$.

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Consider $\mathcal{S}\colon \mathcal{D}_{\mathcal{SO}} \to \mathcal{D}_{\mathcal{SO}}$ defined as

$$\mathcal{S}\mathbf{U} = \mathcal{S}(\mathbf{U}_\rho + \mathbf{U}_\tau + \mathbf{U}_\zeta) := -\mathbf{U}_\rho + \mathbf{U}_\tau - \mathbf{U}_\zeta$$

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Since $\nabla \cdot \mathbf{U} = 0$ for every $\mathbf{U} \in \mathcal{D}_{\mathcal{S}}$,

$$\nabla \times \nabla \times \mathbf{U} = |\mathbf{U}|^4 \mathbf{U}, \quad \mathbf{U} \in \mathcal{D}_{\mathcal{S}}.$$

is equivalent to

$$-\Delta \mathbf{U} = |\mathbf{U}|^4 \mathbf{U}, \quad \mathbf{U} \in \mathcal{D}_{\mathcal{S}}.$$

Moreover, for every $\mathbf{U} \in \mathcal{D}_{\mathcal{S}}$

$$J(\mathbf{U}) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla \mathbf{U}|^2 - \frac{1}{6}|\mathbf{U}|^6 \, dx.$$

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The (scalar) Yamabe Problem

Consider

$$-\Delta u = u^5, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^3),$$
$$-\Delta v + \frac{3}{4} v = v^5, \quad v \in H^1(\mathbb{S}^3).$$

Solutions in 1-to-1 correspondence via the isometric isomorphism

$$v(\xi) = \frac{u\big(\pi(\xi)\big)}{\varphi\big(\pi(\xi)\big)}, \quad \xi \in \mathbb{S}^3,$$
$$u(x) = \varphi(x) v\big(\pi^{-1}(x)\big), \quad x \in \mathbb{R}^3, \tag{2}$$

where $\pi \colon \mathbb{S}^3 \to \mathbb{R}^3 \cup \{\infty\}$ is the stereographic projection and $\varphi(x) = \sqrt{\frac{2}{1+|x|^2}}$.

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$\mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$,

$$\mathbb{S}^3 = \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \;\middle|\; |\xi_1|^2 + |\xi_2|^2 = 1 \right\}.$$

For $g = (g_1, g_2) \in \mathcal{SO}(2)^2$, $v \in H^1(\mathbb{S}^3)$, and $\xi = (\xi_1, \xi_2) \in \mathbb{S}^3$ define $\gamma_g v(\xi) := v(g_1\xi_1, g_2\xi_2)$ and

$$H_{\mathcal{SO}(2\times 2)} := \left\{ v \in H^1(\mathbb{S}^3) \;\middle|\; \gamma_g v = v \; \forall g \in \mathcal{SO}(2)^2 \right\}.$$

Theorem (W. Ding, 1986)

$H_{\mathcal{SO}(2\times 2)} \hookrightarrow\hookrightarrow L^6(\mathbb{S}^3)$.

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The proof

What is γ_g for $u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$? If $v \in H^1(\mathbb{S}^3)$ is defined via (2), then

$$v(\xi) = v(g\xi) \Leftrightarrow \frac{u(x)}{\varphi(x)} = \frac{u\big(\pi\big(g\pi^{-1}(x)\big)\big)}{\varphi\big(\pi\big(g\pi^{-1}(x)\big)\big)}, \quad g = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \in \mathcal{SO}(2)^2.$$

Let $\mathcal{D}_{\mathcal{SO}(2\times 2)} :=$

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Lemma

- $\mathcal{D}_{\mathcal{SO}(2\times 2)} \hookrightarrow\hookrightarrow L^6(\mathbb{R}^3, \mathbb{R}^3)$.
- $\mathcal{D}_{\mathcal{SO}(2\times 2)} \subset \mathcal{D}_{\mathcal{SO}}$.
- *If* $\mathbf{U} \in \mathcal{D}_{\mathcal{SO}(2\times 2)}$, *then* $\mathbf{U}_\rho, \mathbf{U}_\tau, \mathbf{U}_\zeta \in \mathcal{D}_{\mathcal{SO}(2\times 2)}$.
- $\dim Y = \infty$, *where*
  $Y := \{ \mathbf{U} \in \mathcal{D}_{\mathcal{SO}(2\times 2)} \mid \mathcal{S}\mathbf{U} = \mathbf{U} \} = \{ \mathbf{U} \in \mathcal{D}_{\mathcal{SO}(2\times 2)} \mid \mathbf{U} = \mathbf{U}_\tau \}$.

Theorem (P. H. Rabinowitz, 1986)

E Banach Space, $\dim E = \infty$, $I \in \mathcal{C}^1(E)$ even, $I(0) = 0$ and satisfying (PS) at every positive level, $\inf_{S_r} I > 0$ for some $r > 0$. Assume for every finite-dimensional subspace $F \subset E$ there exists $R(F) > 0$ such that $\sup_{F \setminus B_{R(F)}} I \le 0$. Then there exists $x_n \in E$ such that $I'(x_n) = 0$ and $I(x_n) \to \infty$.

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There exists $\mathbf{U}_n \in Y$ *such that* $\mathbf{U}_n$ *is a solution to* (1) *and* $J(\mathbf{U}_n) \to \infty$.

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