Alternating links, rational balls & tilings

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(on jt. work with Josh Greene)

1] Questions

$$S^3 = \partial B^4$$

$$\Sigma_2(S^3, \text{unknot}) = \Sigma_2(B^4, \text{unknotted disk})$$

Generalise:

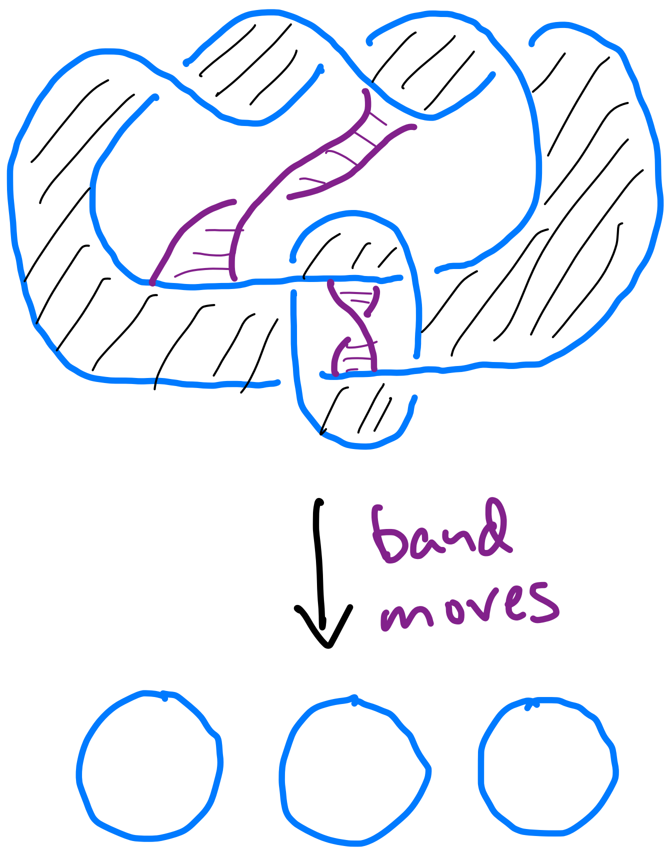
- Which rational homology 3-spheres Y bound rational homology 4-balls W?
- Which knots in S^3 are slice?
- Which links in S^3 bound surfaces $F \hookrightarrow B^4$, $\chi(F) = 1$?

$W = \Sigma_2(B^4, F)$ is $\mathbb{Q}HB$

[Everything smooth]

GOAL: classify alternating links with $\Sigma_2(S^3, L) = \partial\, \mathbb{Q}HB$

2) <u>Example</u>



L alternating link

Λ_b "black lattice" has rank 2

Gram matrix $\begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$, det $16 = 4^2$

L bounds $F = \text{disk} \amalg \text{Möb. band}$

$\chi(F) = 1$

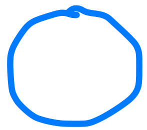
$$\Sigma_2(S^3, L) = \partial \Sigma_2(B^4, F) \quad \mathbb{Q}HB$$

3

Theorem (Greene–O.)

Let L be an alternating link s.t. Λ_b has rank n and determinant 2^n.

TFAE:

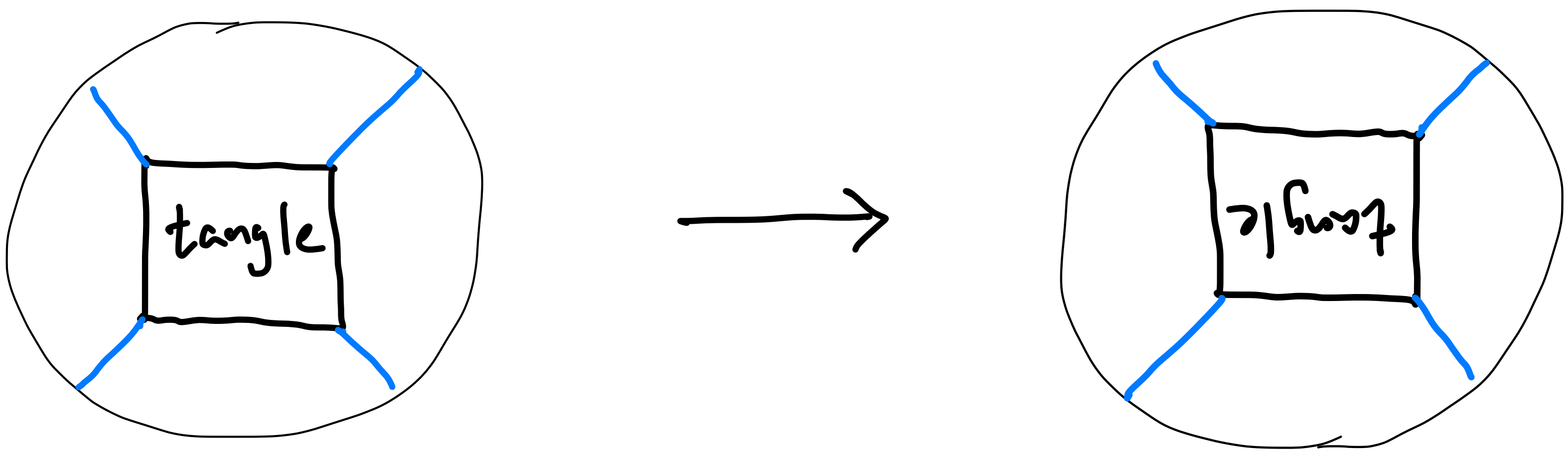
1. $\Sigma_2(S^3, L)$ bounds a $\mathbb{Q}HB$;
2. Λ_b is a 2-cube tiling lattice;
3. L is expanded from the crossingless unknot  by a finite sequence of moves I and II;
4. L may be converted to the k-component unlink by a sequence of $(k-1)$ band moves and finitely many Conway mutations for some $k \in \mathbb{N}$.

4]

2-cube tiling lattice:

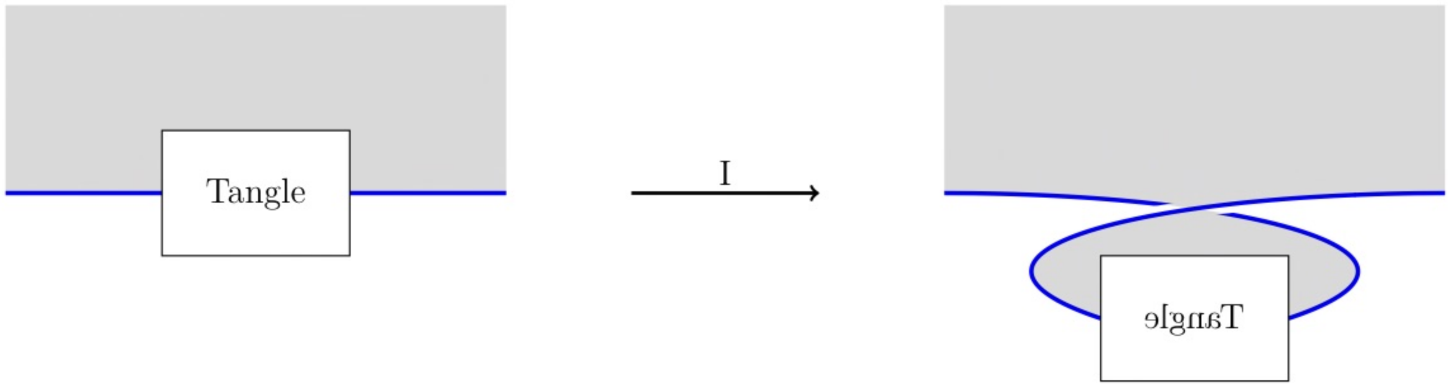
a lattice $\Lambda \subset \mathbb{Z}^n$ whose vertices are centres of cubes of side 2 which tile $\mathbb{R}^n$

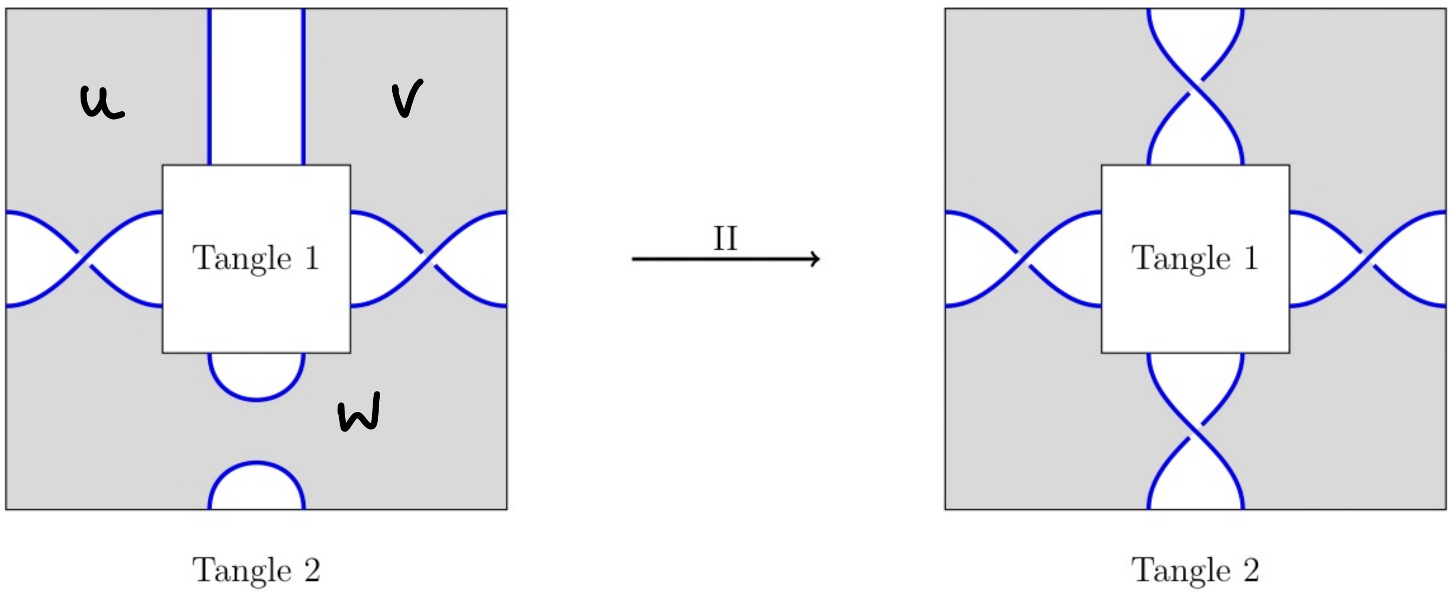
Conway mutation:



(suffices for our needs)

5 Moves I and II:



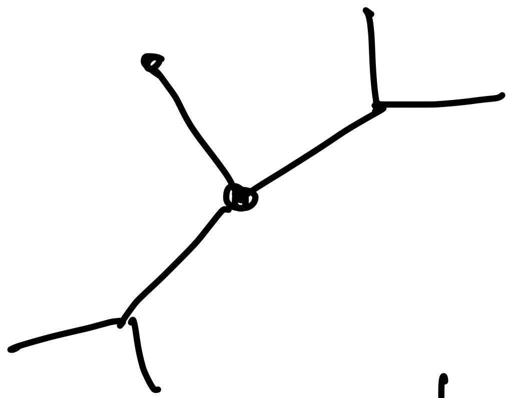


(require u, v in different components of black graph \ {w})

6) Lots of examples

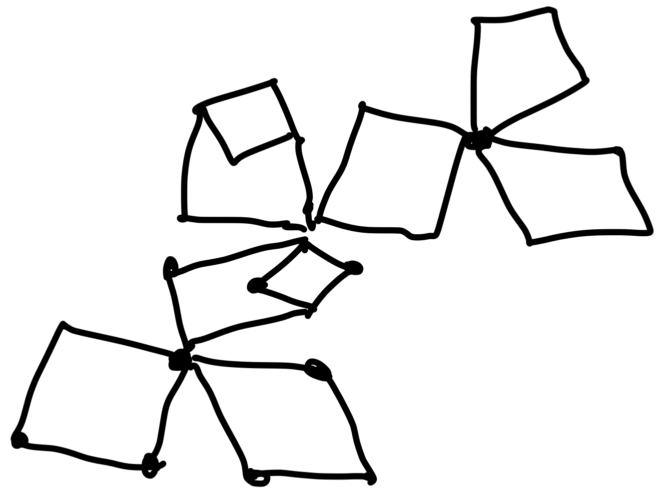
Black graph of unknot

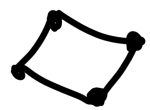
$\xrightarrow{\text{moves I}}$



any tree

$\xrightarrow[\text{\& II}]{\text{moves I}}$



Replace any edge with 

7 Proof sketch

① $\Sigma_2(S^3, L)$ bounds a $\mathbb{Q}$HB;

$$\Downarrow$$

Donaldson's diagonalisation thm + Heegaard Floer correction terms (à la Greene-Jabuka)

② Λ_b is a 2-cube tiling lattice;

③ L is expanded from the crossingless unknot $\bigcirc$ by a finite sequence of moves I and II;

④ L may be converted to the k-component unlink by a sequence of $(k-1)$ band moves and finitely many Conway mutations for some $k \in \mathbb{N}$.

8] Proof sketch

① $\Sigma_2(S^3, L)$ bounds a $\mathbb{Q}$HB ;

② Λ_b is a 2-cube tiling lattice ;

$\Downarrow$ uses Minkowski conjecture (1896) proved by Hajós (1941): every tiling of $\mathbb{R}^n$ by cubes has a pair of cubes which share a facet

③ L is expanded from the crossingless unknot $\bigcirc$ by a finite sequence of moves I and II ;

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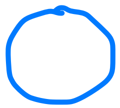
③ L is expanded from the crossingless unknot  by a finite sequence of moves I and II;

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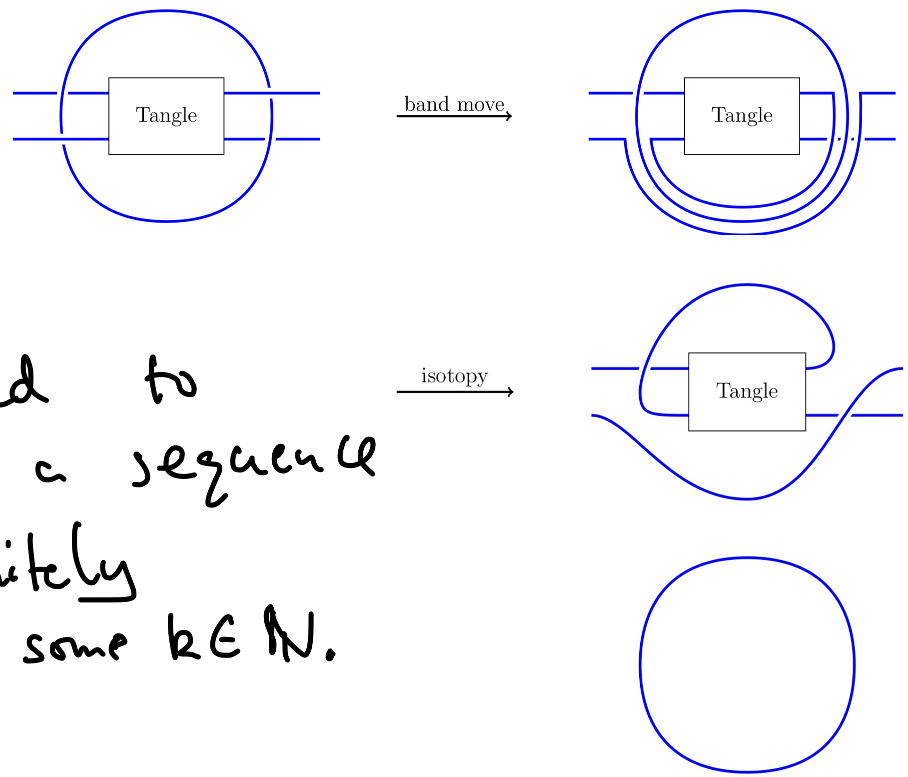

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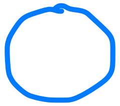
③ L is expanded from the crossingless unknot  by a finite sequence of moves I and II;

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④ $\Rightarrow$ ① : Take double branched cover

Thanks!