TORUS-LIKE SOLUTIONS FOR THE LANDAU-DE GENNES MODEL

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nematic liquid crystals: rod-like molecules filling $\Omega \subseteq \mathbb{R}^3$ (smooth, bd) with (mean) orientational order

Landau-De Gennes Theory

$\Omega \ni x \longrightarrow Q(x) \in \mathcal{S}_0 \subseteq M_{3\times 3}(\mathbb{R})$ **Q-Tensor field**

$\mathcal{S}_0 = \{A : A = A^t,\ \operatorname{Tr} A = 0\} \simeq \mathbb{R}^5$

$$\boxed{\tilde{\beta}(Q) = \sqrt{6}\,\frac{\operatorname{Tr} Q^3}{|Q|^3},\ Q \neq 0}$$

signed biaxiality parameter, $\tilde{\beta}(\cdot) \in [-1, 1]$

$Q(x) \in \mathcal{S}_0$, $\sigma(Q(x)) = \{\lambda_1(x) \leq \lambda_2(x) \leq \lambda_3(x)\}$, $\sum_{j=1}^{3} \lambda_j(x) \equiv 0$

- $\{x \mid Q(x) = 0\} = \{\lambda_1 = \lambda_2 = \lambda_3\} \subseteq \Omega$ **isotropic phase**
- $\{x \mid \tilde{\beta}(Q(x)) = \pm 1\} \subseteq \Omega$ pos/ $\lambda_1 = \lambda_2$ neg/ $\lambda_2 = \lambda_3$ **uniaxial phase** $\Rightarrow$ $Q = s(n \otimes n - \frac{1}{3}\mathrm{Id})$, $s \in \mathbb{R}$, $n \in S^2$
- $\{x \mid \tilde{\beta}(Q(x)) = t\} = \{\tilde{\beta} = t\} \subseteq \Omega$, $t \in [-1, 1]$ **biaxial surfaces**

$Q \in W^{1,2}(\Omega; \mathcal{S}_0)$, $\mathcal{F}_{LG} : W^{1,2}(\Omega; \mathcal{S}_0) \longrightarrow \mathbb{R}$ **LDG energy functional**

$$\boxed{\mathcal{F}_{LG}(Q;\Omega) = \int_\Omega \frac{1}{2}|\nabla Q|^2 + \frac{1}{L} F(Q)\,dx}$$

1-const approximation for the **elastic energy**, $L > 0$

$$\boxed{F(Q) = -\frac{a^2}{2}|Q|^2 - \frac{b^2}{3}\operatorname{Tr} Q^3 + \frac{c^2}{4}|Q|^4}$$

$F : \mathcal{S}_0 \to \mathbb{R}$ **bulk potential**

a, b, c material depend. const.

$\tilde{F} := F - \min_{\mathcal{S}_0} F \geq 0$, $Q_{min} = \{\tilde{F} = 0\} \subseteq \mathcal{S}_0$ **Vacuum manifold**

$Q_{min} \subseteq \{|Q| = \sqrt{\tfrac{2}{3}}\, s_+\} \simeq S^4$, $s_+(a,b,c) = \dfrac{b^2 + \sqrt{b^4 + 24a^2c^2}}{4c^2}$

$Q_{min} = \{s_+(n \otimes n - \frac{1}{3}\mathrm{Id}),\ n \in S^2\} \subseteq \{\tilde{\beta} = 1\}$, $Q_{min} \simeq \mathbb{R}P^2$

$\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$

$\pi_2(\mathbb{R}P^2) = \mathbb{Z}$

nontrivial Topology $\Longrightarrow$ **Topological defects**

Rescaling $Q \rightsquigarrow s_+\sqrt{\tfrac{2}{3}}\,Q$, so that $Q_{min} = \mathbb{RP}^2 \subseteq S^4$

outher normal

$\partial\Omega \ni x \longrightarrow Q_b(x) = \sqrt{\tfrac{3}{2}}\left(\vec{n}(x)\otimes\vec{n}(x) - \tfrac{1}{3}\mathrm{Id}\right) \in Q_{min}$ radial anchoring

$\mathcal{A}(\Omega;\mathcal{S}_0) = W^{1,2}_{Q_b}(\Omega;\mathcal{S}_0)$ admissible configurations

$$\boxed{\frac{3}{2L s_+^2}\tilde{F}\left(s_+\sqrt{\tfrac{2}{3}}Q\right) = \mu\underbrace{\left(\frac{|Q|^2-1}{2}\right)^2}_{\geq 0} + \lambda\underbrace{\left(\frac{|Q|^4}{4\sqrt{6}} - \frac{\mathrm{tr}\,Q^3}{3} + \frac{1}{12\sqrt{6}}\right)}_{=:\,W(Q)\,\geq\, 0}}$$

$\mu(a,b,c,L)$ nematic correlation lenght

$\lambda(a,b,c,L)$ biaxial coherence lenght

Case I: $Q\in\mathcal{A}(\Omega;S^4)$, i.e., let $|Q|\equiv 1$, Lyuksyutov constraint

$$\mathcal{E}_\lambda(Q) = \int_\Omega \frac{1}{2}|\nabla Q|^2 + \lambda W(Q)\,dx$$

$$\boxed{\Delta Q + |\nabla Q|^2 Q = \lambda\nabla_T W(Q)} \quad (EL)$$

Case II: $Q\in\mathcal{A}(\Omega;\mathcal{S}_0)$ and $\mu\to\infty$

$(\Rightarrow |Q|\sim 1)$ Lyuksyutov regime

$$\mathcal{F}^\mu_\lambda(Q) = \int_\Omega \frac{1}{2}|\nabla Q|^2 + \mu\left(\frac{|Q|^2-1}{2}\right)^2 + \lambda W(Q)\,dx$$

$$\boxed{\Delta Q + \mu(1-|Q|^2)Q = \lambda\nabla W(Q)} \quad (EL)$$

THM (Dipasquale-Millot-P.; ARMA '21) For $\partial\Omega\in C^3$, $\Omega\simeq$ ball

1) minimizers Q_λ, resp. Q^μ_λ, are smooth (~Schoen-Uhlenbeck), i.e., in $C^2(\bar{\Omega})\cap C^\omega(\Omega)$

2) The biaxiality surfaces $\{\tilde{\beta}\circ Q_\lambda = t\}\neq\emptyset$, resp. $\{\tilde{\beta}\circ Q^\mu_\lambda = t\}\neq\emptyset$ ($Q^\mu_\lambda \neq 0$), $t\in(-1,1)$ regular, exibit nontrivial topology (positive genus, nonsimply connected)

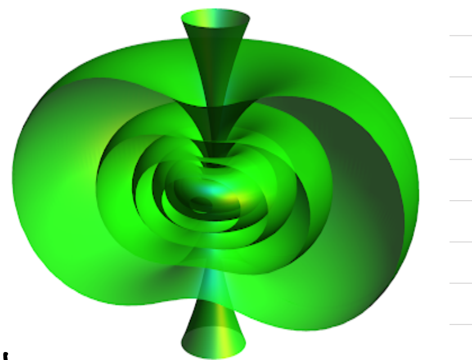
Cor: The nematic hedgehog in $\Omega = \{|x|<1\}$

$$H^\mu_\lambda(x) = s^\mu_\lambda(|x|)\left(\frac{x}{|x|}\otimes\frac{x}{|x|} - \frac{1}{3}\mathrm{Id}\right), \quad s^\mu_\lambda : [0,1]\to\left[0,\sqrt{\tfrac{3}{2}}\right]$$

is uniaxial, nonminimizing and unstable

CJ Q_λ and Q^μ_λ are axially symmetric

The biaxial surfaces are tori ($\sim S^3 \to S^2$ Hopf fibration)



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Gain insight through axial symmetry reduction

1) Domains: $S^1 \circlearrowright \mathbb{R}^3$, $x=(x',x_3) \xrightarrow{R\in S^1} (Rx',x_3)=:R\cdot x$, $R\Omega=\Omega \quad \forall R\in S^1$

2) Tensors: $S^1 \circlearrowright \mathcal{S}_0$, $Q \xrightarrow{R\in S^1} RQR^{-1}=:R\cdot Q$, $\mathcal{S}_0 = L_0\oplus L_1\oplus L_2 \simeq \mathbb{R}\oplus\mathbb{C}\oplus\mathbb{C}$

3) maps: $\mathcal{A}^{sym}(\Omega;\mathcal{S}_0)$ (or S^4) admissible + S^1-equivariant, i.e., $\boxed{Q(Rx)=RQ(x)R^{-1} \text{ a.e.}}$

with S^1-equivariant boundary values (e.g. the radial anchoring)

Symmetric criticality (~ Palais): Q_λ min. of $\mathcal{E}_\lambda$ (or $\mathcal{F}_\lambda^\mu$) in $\mathcal{A}^{sym}_{Q_b}(\Omega;S^4)$ (or $\mathcal{S}_0$) is a weak sol to (CEL) (or (EL))

Symmetry + 2D regularity $\Rightarrow$ smoothness if $x'\neq 0$ for (CEL),
— — everywhere for (EL)

THM (Dipasquale-Millot-P.; Submitted)

1) minimizers $Q_\lambda\in\mathcal{A}^{sym}_{Q_b}(\Omega;S^4)$ are in $C^2(\overline{\Omega}\setminus\Sigma)\cap C^\omega(\Omega\setminus\Sigma)$
$\Sigma\subseteq\Omega\cap\{x'=0\}$ empty or finite.

2) $\forall p\in\Sigma \quad \exists\,\Phi\in C^\omega(S^2;S^4)$ harmonic S^1-equivariant s.t. $\lim_{\sigma\to0} Q_\lambda(p+\sigma x)=\Phi\left(\frac{x}{|x|}\right)$,

$$S^2\simeq\mathbb{CP}^1 \ni \boxed{[z_0,z_1] \xrightarrow{\Phi} \pm\left(\frac{|z_0|^2-|z_1|^2}{|z_0|^2+|z_1|^2}, \frac{2\bar z_0 z_1 e^{i\theta}}{|z_1|^2+|z_0|^2}, 0\right)}\in S^4\subseteq\mathbb{R}\oplus\mathbb{C}\oplus\mathbb{C}, \quad e^{i\theta}\in S^1$$

equatorial embedding

RMK: Classification of Φ in terms of its twistor lift (Calabi)

$\tilde\Phi:\mathbb{CP}^1\to\mathbb{CP}^3$

$$\boxed{\Phi=\pm\tau\circ\tilde\Phi}$$

$$\begin{tikzcd} & \mathbb{CP}^3\simeq SO(5)/U(2) \arrow[d, "\tau"] \\ S^2\simeq\mathbb{CP}^1 \arrow[ur, "\tilde\Phi"] \arrow[r, "\Phi"'] & S^4 \end{tikzcd}$$

twistor fibration

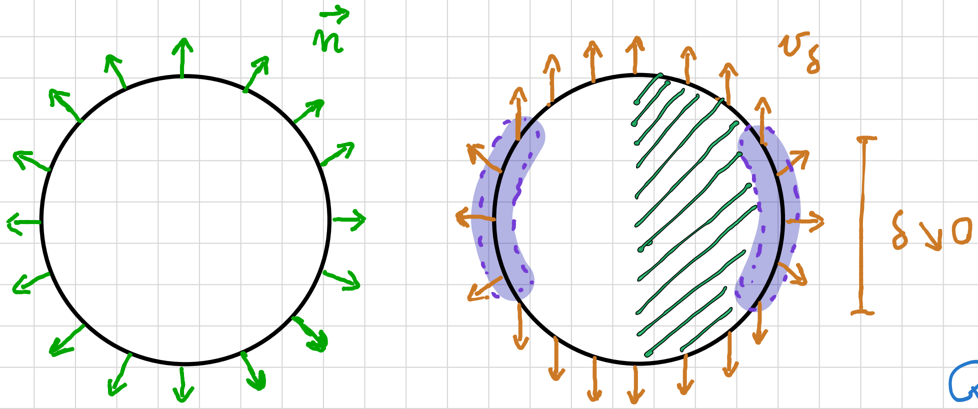
$$\boxed{\tilde\Phi([z_0,z_1])=\left[z_0^3, \mu_1 z_0^2 z_1, \mu_2 z_0 z_1^2, -\frac{\mu_1\mu_2}{3}z_1^3\right], \quad (\mu_1,\mu_2)\in\mathbb{C}^2}$$

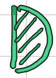
minimality $\Rightarrow \mu_1=e^{i\theta}$, $\mu_2=0$

Construction of Torus-like minimizers

$\Omega = \{|x|<1\}$, $Q_b^\delta(x) = \sqrt{\frac{3}{2}}\left(v_\delta(x)\otimes v_\delta(x) - \frac{1}{3}\mathrm{Id}\right)$, $Q^\delta \in \mathcal{A}^{\mathrm{Sym}}_{Q_b^\delta}(\Omega;S^4)$

$v_\delta \in C^\infty(S^2;S^2)$ S^1-equivariant, $v_\delta \in [\vec{n}]$ homotopic, $0<\delta\ll 1$



$D^+ =$  $\subseteq \Omega\cap\{x_2=0,\ x_1>0\}$

$Q_b^\delta \in W^{1/2,2}(\partial D^+;\mathbb{RP}^2)$

$[Q_b^\delta]\neq 0$ in $\pi_1(\mathbb{RP}^2)=\mathbb{Z}_2$

$Q_b^\delta \xrightarrow{\delta\searrow 0} e_0 = Q_b[v_0]$ | $W^{1/2,2}$-bubbling along the equator

THM (Dipasquale-Millot-P.; Submitted)

1) For $0<\delta\ll 1$, minimizers $Q^\delta \in \mathcal{A}^{\mathrm{Sym}}_{Q_b^\delta}(\Omega;S^4)$ are smooth in $\overline{\Omega}$

2) Except finitely many $t\in[-1,1]$, $\{\tilde\beta\circ Q^\delta = t\}$ is a finite union of Tori

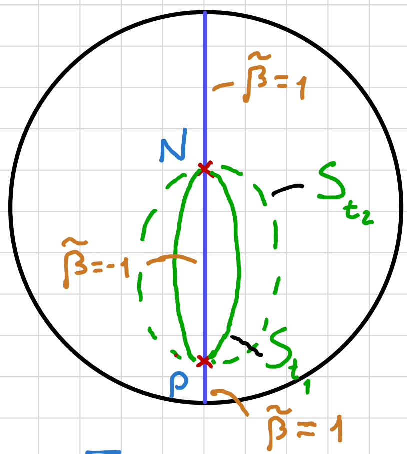
Construction of split minimizers $\Omega=\{|x|<1\}$

$Q\in\mathcal{A}^{\mathrm{Sym}}_{Q_b^\delta}(\Omega;S^4)$, $Q_b^\delta(x) = \tau\circ\tilde\Phi(x)$ harmonic sphere, $\mu_1\in S^1$, $\mu_2=\delta\searrow 0$ || $W^{1,2}$-bubbling at the south pole

THM (Dipasquale-Millot-P.; Submitted)

1) For $0<\delta\ll 1$, minimizers $Q^\delta \in \mathcal{A}^{\mathrm{Sym}}_{Q_b^\delta}(\Omega;S^4)$ are singular in $\overline{\Omega}$, $\Sigma = \mathrm{Sing}\,Q^\delta \neq \emptyset$

2) When $\Sigma\neq\emptyset$ and $t\in(-1,\bar{t}(\delta))$ is regular, Then $S_t = \{\tilde\beta\circ Q^\delta = t\}$ contains invariant spheres with poles on Σ



$N,P\in\Sigma$, $-1<t_1<t_2<\bar{t}(\delta)$

$\bar{t}(\delta) := \min_{\partial\Omega}\tilde\beta\circ Q_b^\delta$