Noncommutativity in an algebraic theory of clones

Antonino Salibra
Università Ca'Foscari Venezia

in collaboration with

Antonio Bucciarelli
Université de Paris

8th European Congress of Mathematics

https://arxiv.org/abs/2010.14461v2

Summary of the talk

1. Clones as many-sorted algebras

2. How a clone becomes a (one-sorted) clone algebra

3. How a free algebra becomes a clone algebra

4. The variety of clone algebras as a one-sorted algebraic theory of clones (with composition) and free algebras (with substitution)

5. Functional clone algebras and the Representation Theorem

6. The lattices of equational theories and our answer to the Birkhoff-Maltsev Problem

7. Noncommutativity in clone algebras

8. To study the category of all varieties of algebras through the variety of pure clone algebras

Clones as many-sorted algebras

- The term operations of an algebra constitute a clone. Important properties of an algebra, for example, subalgebras and congruences, depend on the clone of its term operations.

- Let $\mathbf{A}$ be an algebra of type τ. A *clone* $C = \bigcup_{n \geq 0} C_n$ on $\mathbf{A}$ is a many-sorted set of finitary operations on A that contains the projections and is closed under composition:

  (a) $\pi_i^n \in C_n$; $\qquad \pi_i^n(a_1, \ldots, a_n) = a_i$.

  (b) If $f \in C_n$, $g_1, \ldots, g_n \in C_k$, then $f(g_1, \ldots, g_n) \in C_k$:

  $$f(g_1, \ldots, g_n)(a_1, \ldots, a_k) = f(g_1(a_1, \ldots, a_k), \ldots, g_n(a_1, \ldots, a_k)).$$

  (c) If $\sigma \in \tau$ has arity k, then $\sigma^{\mathbf{A}} \in C_n$.

- Composition and Projections are many-sorted operations.

How a clone becomes a clone algebra I

- Clones become one-sorted clone algebras if the finitary operations "loose" their arity.

- The *top expansion* of an n-ary operation $f : A^n \to A$ is the infinitary operation $f^\top : A^\omega \to A$ defined by

$$f^\top(s) = f(s_1, \dots, s_n), \quad \text{for all } s \in A^\omega.$$

  $f^\top$ extends the finitary operation f by countably many dummy arguments.

- The kernel of top operator defines an equivalence relation:

$$f \approx_A g \text{ iff } f^\top = g^\top$$

  f and g are in the same block w.r.t. $\approx_A$ if they are equal up to dummy arguments in the last arguments.

How a clone becomes a clone algebra II

- All projections π_i^n $(n \geq i)$ on i-argument are $\approx_A$-equivalent.

- A *$\approx_A$-block* $B = \{f_1 < f_2 < \cdots < f_n < \dots\}$ is totally ordered by the arities of the operations f_i. The arity of the block B is the arity of the minimum element f_1.

  We define the top expansion $B^\top$ of a block B as

$$B^\top = f_1^\top = \cdots = f_n^\top = \dots$$

- Every clone C is union of blocks

How a clone becomes a clone algebra III

- The top expansion $C^\top = \{f^\top : \ f \in C\}$ of a clone C on a τ-algebra $\mathbf{A}$ contains the projections $\mathsf{e}_i^\omega : A^\omega \to A$, defined by

$$\mathsf{e}_i^\omega(s) = s_i \qquad (\mathsf{e}_i^\omega = (\pi_i^n)^\top \text{ for every } n \geq i),$$

  it is closed under the **one-sorted** n-ary composition acting on the first n coordinates. Let $s = (s_1, s_2, \dots, s_n, s_{n+1}, \dots) \in A^\omega$.

$$q_n^\omega(f^\top, g_1^\top, \dots, g_n^\top)(s) = f^\top(g_1^\top(s), \dots, g_n^\top(s), s_{n+1}, \dots),$$

  and it is closed under the τ-operations:

$$\sigma^\omega(g_1^\top, \dots, g_n^\top)(s) = \sigma^{\mathbf{A}}(g_1^\top(s), \dots, g_n^\top(s)).$$

  $\mathbf{C}^\top = (C^\top, q_n^\omega, \mathsf{e}_i^\omega, \sigma^\omega)_{n\geq 0, i \geq 1, \sigma\in\tau}$ is called a *block algebra on the τ-algebra* $\mathbf{A}$.

- **Proposition.** There is a bijective correspondence between clones on $\mathbf{A}$ and block algebras on $\mathbf{A}$.

How a free algebra with substitutions is a clone algebra

- Let $\mathcal{V}$ be a variety of type τ and $\mathbf{F}_\mathcal{V}$ be the free $\mathcal{V}$-algebra over $\{v_1, v_2, \dots, v_n, \dots\}$

- **Definition.** An endomorphism f of $\mathbf{F}_\mathcal{V}$ is *n-finite* if $f(v_i) = v_i$ for $i > n$.

- The set of all n-finite endomorphisms can be collectively expressed by an $(n+1)$-ary operation $q_n^\mathbf{F}$ of **substitution** on $\mathbf{F}_\mathcal{V}$:

$$q_n^\mathbf{F}(t, u_1, \dots, u_n) = f(t), \quad \text{for every } t, u_1, \dots, u_n \in F_\mathcal{V}, \tag{1}$$

  where f is the unique n-finite endomorphism of $\mathbf{F}_\mathcal{V}$ which sends the generator v_i to u_i $(1 \le i \le n)$.

- Then the algebra $\mathbf{Clone}(\mathbf{F}_\mathcal{V}) = (\mathbf{F}_\mathcal{V}, q_n^\mathbf{F}, \mathsf{e}_i^\mathbf{F})$, where $\mathsf{e}_i^\mathbf{F} = v_i \in I$, is called *the clone $\mathcal{V}$-algebra* associated to the free algebra of $\mathcal{V}$.

- The algebra $\mathbf{Clone}(\mathbf{F}_\mathcal{V}) = (\mathbf{F}_\mathcal{V}, q_n^\mathbf{F}, \mathsf{e}_i^\mathbf{F})$ is isomorphic to a minimal block algebra of type τ.

- There is a bijective correspondence between free algebras with substitutions and minimal block algebras.

Clone algebras

- Given an algebraic type τ, the type of clone τ-algebras is $\tau \cup \{q_n : n \geq 0\} \cup \{\mathsf{e}_i : i \geq 1\}$

Definition. A *clone τ-algebra* (CA$_\tau$) is an algebra $\mathbf{C} = (\mathbf{C}_\tau, q_n^{\mathbf{C}}, \mathsf{e}_i^{\mathbf{C}})_{n\geq 0, i\geq 1}$ satisfying the following conditions:

(C0) $\mathbf{C}_\tau = (C, \sigma^{\mathbf{C}})_{\sigma\in\tau}$ is a τ-algebra;

(C1) $q_n(\mathsf{e}_i, x_1, \dots, x_n) = x_i$ $(1 \leq i \leq n)$;

(C2) $q_n(\mathsf{e}_j, x_1, \dots, x_n) = \mathsf{e}_j$ $(j > n)$;

(C3) $q_n(x, \mathsf{e}_1, \dots, \mathsf{e}_n) = x$ $(n \geq 0)$;

(C4) $q_n(x, y_1, \dots, y_n) = q_k(x, y_1, \dots, y_n, \mathsf{e}_{n+1}, \dots, \mathsf{e}_k)$ $(k > n)$;

(C5) $q_n(q_n(x, y_1, \dots, y_n), z_1, \dots, z_n) = q_n(x, q_n(y_1, z_1, \dots, z_n), \dots, q_n(y_n, z_1, \dots, z_n))$;

(C6) $q_n(\sigma(x_1, \dots, x_k), y_1, \dots, y_n) = \sigma(q_n(x_1, y_1, \dots, y_n), \dots, q_n(x_k, y_1, \dots, y_n))$ for every $\sigma \in \tau$.

Theorem. CA$_\tau = \mathbb{HSP}(\mathsf{BLK}_\tau)$.

The proof is not trivial.

The clone of representable operations

Let $\mathbf{C}$ be a clone τ-algebra.

- *An operation $f : C^k \to C$ is $\mathbf{C}$-representable* if there exists $a \in C$ such that
$$f(b_1, \dots, b_k) = q_k(a, b_1, \dots, b_k), \text{ for every } b_1, \dots, b_k \in C.$$
We have $a = f(\mathsf{e}_1, \dots, \mathsf{e}_k)$.

- **Theorem.** The set $R(\mathbf{C})$ of the $\mathbf{C}$-representable operations is a clone on $\mathbf{C}_\tau$ such that
$$\text{Block algebra } R(\mathbf{C})^\top \hookrightarrow \mathbf{C}$$

Definition. A clone algebra $\mathbf{C}$ is said to be *finite-dimensional* if
$$\text{Block algebra } R(\mathbf{C})^\top \cong \mathbf{C}$$

Corollary. $\mathsf{Fi\,CA}_\tau = \mathbb{I}\,\mathsf{BLK}_\tau$

Functional clone algebras

- Clones (of finitary operations) $\Leftrightarrow$ Block algebras (of infinitary operations)

- ω-Clones (of infinitary operations) $\Longleftrightarrow$ Clone algebras

- **Definition.** A *functional clone algebra* (FCA) is a set of infinitary operations from A^ω into A containing the projections π_i^ω and closed under the n-ary composition acting on the first n coordinates:

$$q_n^\omega(\varphi, \psi_1, \dots, \psi_n)(s) = \varphi(\psi_1(s), \dots, \psi_n(s), s_{n+1}, s_{n+2}, \dots)$$

  If A is a τ-algebra we extend every basic operation $\sigma^A : A^n \to A$ to $\sigma^\omega : A^\omega \to A$:

$$\sigma^\omega(\psi_1, \dots, \psi_n)(s) = \sigma^A(\psi_1(s), \dots, \psi_n(s)).$$

- Notice that $\sigma^\omega(\psi_1, \dots, \psi_n) = q_n^\omega((\sigma^A)^\top, \psi_1, \dots, \psi_n)$ and $(\sigma^A)^\top = \sigma^\omega(\pi_1^\omega, \dots, \pi_n^\omega)$.

- $\mathsf{BLK}_\tau \subseteq \mathsf{FCA}_\tau \subseteq \mathsf{CA}_\tau$.

- Question: Does the abstract notion of CA coincide with the concrete notion of FCA?

The representation Theorem

- A CA is functionally representable if it is isomorphic to a FCA.

- **Theorem.** $\mathsf{CA} = \mathbb{I}\,\mathsf{FCA} = \mathbb{HSP}(\mathsf{BLK})$.

- **Corollary.** Every ω-clone is algebraically generated by a family of clones by using direct products, subalgebras and homomorphic images.

- The following diagram provides an outline of the proof:

CA	$=$	$\mathbb{I}\,$RCA	Each CA is isomorphic to a point relativized CA
	$\subseteq$	$\mathbb{I}\,\mathbb{S}\,\mathbb{U}_p\,$FCA	Each RCA embeds into an ultrapower of a FCA
	$\subseteq$	$\mathbb{I}\,\mathbb{S}\,\mathbb{P}\,$FCA	Each ultrapower of a FCA is isomorphic to a subdirect product of a family of FCAs.
	$\subseteq$	$\mathbb{I}\,$FCA	FCAs are closed under subalgebras and direct products
	$\subseteq$	CA	FCAs are clone algebras

The lattices of equational theories

- A *lattice of equational theories* is the lattice $L(T)$ of all equational theories containing some given equational theory T. One of the members of the lattice of all equational theories of groups is the equational theory of Abelian groups.

- The lattice $L(T)$ is algebraic and coatomic, with a compact top element Lampe (1986) has shown that $L(T)$ obeys the Zipper condition (a nontrivial implication in the language of lattices).

- Problem (Birkhoff-Maltsev): **Find an algebraic characterisation of the lattices of equational theories**.

- Characterise the lattices of equational theories as:

  - Newrly 1993 and Nurakunov 2008 have characterised the lattices of equational theories as the congruence lattices of suitable class of algebras. The variety generated by these classes is unknown.

Theorem. A lattice L is isomorphic to a lattice of equational theories if and only if L is isomorphic to a congruence lattice of a finite-dimensional clone algebra.

Noncommutativity in clone algebras

- **Definition.** An element c of a clone algebra $\mathbf{C}$ is n-*central* if $\theta(c, \mathsf{e}_1), \dots, \theta(c, \mathsf{e}_n)$ are a tuple of complementary factor congruences on $\mathbf{C}$:

  $$\mathbf{C} \cong \mathbf{C}/\theta(c, \mathsf{e}_1) \times \cdots \times \mathbf{C}/\theta(c, \mathsf{e}_n).$$

  We say that c is *central* if it is n-central for some n.

- It is possible to equationally characterise n-central elements through q_n and $\mathsf{e}_1, \dots, \mathsf{e}_n$.

  Theorem. (1) A central element of $\mathbf{C}$ is finite-dimensional.

  (2) The set of central elements is a pure subalgebra of $\mathbf{C}$.

Noncommutativity in clone algebras II

Theorem. Let $\mathbf{C}$ be a clone algebra and let $B = CE_n(\mathbf{C})$ be the set of n-central elements of $\mathbf{C}$. For every $i \leq n$, we define

$$t_i^n(x, y, z) = q_n(x, y, \dots, y, z, y, \dots, y) \quad z \text{ at position } i$$

$$x \cap_i^n y = q(x, t_1(y, \mathsf{e}_i, x), t_2(y, \mathsf{e}_i, x), \dots, t_n(y, \mathsf{e}_i, x))$$

- For every $1 \leq i \leq n$, $S_i^n(B) = (B, t_i^n, \cap_i^n, \mathsf{e}_1, \dots, \mathsf{e}_n)$ is (term equivalent to) a skew Boolean algebra with intersection such that e_i is the bottom element and e_j $(j \neq i)$ is maximal.

- Every permutation σ of the symmetric group S_n determines a bunch of isomorphisms

$$S_1^n(B) \cong S_{\sigma 1}^n(B); \dots \qquad \dots S_n^n(B) \cong S_{\sigma n}^n(B)$$

  which shows the inner symmetry of the n-central elements.

- q_n can be recovered by the t_i^n and $\cap_i^n$ because

$$q_n(x, y_1, \dots, y_n) = t_1(x, t_2(x, t_3(x, \dots t_{n-1}(x, y_n, y_{n-1}) \dots, , y_3), y_2), y_1).$$

The category of all varieties (of any type) I

(1) The importance of the variety CA_0 of pure CAs in the study of varieties

(2) We study the category of all varieties through a category of CAs. Homomorphisms of CAs are easier to manage than interpretations of varieties.

- When is a CA_τ the free algebra of a variety of type τ?
- A CA_τ $\mathbf{C}$ is *minimal* if $\mathbf{C}$ has no proper subalgebras.
- **Theorem.** Let $\mathbf{C} = (\mathbf{C}_\tau, q_n^{\mathbf{C}}, \mathsf{e}_i^{\mathbf{C}})$ be a minimal CA_τ. Then

  (i) $\mathbf{C}_\tau$ is the free algebra in the variety $\mathsf{Var}(\mathbf{C}_\tau)$;

  (ii) $\mathbf{C}$ is the free algebra over an empty set of generators in $\mathsf{Var}(\mathbf{C})$.

  Corollary. A CA_τ $\mathbf{C}$ is minimal if and only if $\mathbf{C} \cong \mathbf{Cl}(\mathcal{V})$ for some variety $\mathcal{V}$ of type τ.

• The category of all varieties II

- A *pure homomorphism* from a CA$_\tau$ $\mathbf{C}$ into a CA$_\nu$ $\mathbf{D}$ is a homomorphism $f : \mathbf{C}_0 \to \mathbf{D}_0$ of pure reducts

- Why pure homomorphisms?
  - Let $\mathcal{V}$ be a τ-variety and $\mathcal{W}$ be a ν-variety. If $\mathbf{Cl}(\mathbf{F}_\mathcal{V})$ and $\mathbf{Cl}(\mathbf{F}_\mathcal{W})$ are purely isomorphic then
    (i) the theories of $\mathcal{V}$ and $\mathcal{W}$ determine isomorphic clones;
    (ii) the varieties $\mathcal{V}$ and $\mathcal{W}$ are categorically isomorphic
    (Example: the varieties of Boolean algebras and Boolean rings).
  - (Strong Maltsev conditions) Let $\mathcal{V}$ be a variety with a unique ternary operator p satisfying the Maltsev identities $p(x, y, y) = x$ and $p(x, x, y) = y$. A variety $\mathcal{W}$ is permutable iff there is a pure homomorphism from $\mathbf{Cl}(\mathbf{F}_\mathcal{V})$ into $\mathbf{Cl}(\mathbf{F}_\mathcal{W})$.
  - (The category $\mathcal{VAR}$ of all varieties): An interpretation of a variety $\mathcal{V}$ into a variety $\mathcal{W}$ is a pure homomorphism from $\mathbf{Cl}(\mathbf{F}_\mathcal{V})$ into $\mathbf{Cl}(\mathbf{F}_\mathcal{W})$.

The category of all varieties III

- The category $\mathcal{CA}$ has the class of all CAs as objects and pure homomorphisms as arrows.

- **Proposition.**

  (i) The category $\mathcal{CA}$ is equivalent to the variety CA_0 of pure clone algebras.

  (ii) The skeleton of CA_0 is the lattice of interpretability types of the varieties

- **Theorem.** The categories $\mathcal{VAR}$ of all varieties and $\mathcal{MCA}$ of minimal clone algebras are categorically isomorphic:

$$\begin{array}{rcl} \text{Variety } \mathcal{V} \text{ of } \tau\text{-algebras} & \mapsto & \mathcal{V}\text{-clone } \tau\text{-algebra } \mathbf{Cl}(\mathbf{F}_{\mathcal{V}}) \\ \text{Minimal clone } \tau\text{-algebra } \mathbf{C} & \mapsto & \text{Variety } \mathsf{Var}(\mathbf{C}_\tau) \text{ of } \tau\text{-algebras.} \end{array}$$

  We have

$$\mathcal{V} = \mathsf{Var}(\mathbf{Cl}(\mathbf{F}_{\mathcal{V}})_\tau); \qquad \mathbf{C} \cong \mathbf{Cl}(\mathbf{F}_{\mathsf{Var}(\mathbf{C}_\tau)}).$$

- Hereafter, we identify $\mathcal{VAR}$ and $\mathcal{MCA}$

The category of all varieties IV

- The categorical product of $\mathbf{C}, \mathbf{D} \in \mathcal{MCA}$, denoted by $\mathbf{C} \odot \mathbf{D}$ is defined as follows:
  (1) The pure reduct of $\mathbf{C} \odot \mathbf{D}$ is $\mathbf{C}_0 \times \mathbf{D}_0$;
  (2) The operations of $\mathbf{C} \odot \mathbf{D}$ are all $\mathbf{C}_0 \times \mathbf{D}_0$-representable operations ($\mathbf{C} \odot \mathbf{D}$ is minimal).

- We now consider an application to the independence of varieties.

  Definition. Two subvarieties $\mathcal{V}_1, \mathcal{V}_2$ of a variety $\mathcal{V}$ of type τ are said to be *independent*, if there exists a τ-term $t(v_1, v_2)$ such that $\mathcal{V}_1 \models t(v_1, v_2) = v_1$ and $\mathcal{V}_2 \models t(v_1, v_2) = v_2$.

- The *product of the varieties* $\mathcal{V}_1, \mathcal{V}_2$ is defined as $\mathcal{V}_1 \times \mathcal{V}_2 = \mathbb{I}\{\mathbf{A}_1 \times \mathbf{A}_2 : \mathbf{A}_i \in \mathcal{V}_i\}$. This is not the categorical product!

The category of all varieties V

- **Theorem.** (Grätzer, Lakser and Płonka) Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be independent subvarieties of an arbitrary variety $\mathcal{V}$. Then $\mathcal{V}_1$ and $\mathcal{V}_2$ are disjoint and such that their join $\mathcal{V}_1 \vee \mathcal{V}_2$ (in the lattice of subvarieties of $\mathcal{V}$) is their direct product $\mathcal{V}_1 \times \mathcal{V}_2$.

- The following two theorems improve the above theorem by Grätzer, Lakser and Płonka on independent varieties.

- **Theorem.** Let $\mathbf{C}$ and $\mathbf{D}$ be minimal CA_τs and let $\mathbf{E} = \mathbf{C} \times \mathbf{D}$ be the product of $\mathbf{C}$ and $\mathbf{D}$ in the variety CA_τ. Then the following conditions are equivalent:

  1. $\mathbf{E}$ is minimal ($\mathbf{E}$ is canonically isomorphic to $\mathbf{C} \odot \mathbf{D}$ in category $\mathcal{MCA}$).
  2. $\mathsf{Var}(\mathbf{C}_\tau)$ and $\mathsf{Var}(\mathbf{D}_\tau)$ are independent.

  If $\mathbf{E}$ is minimal then $\mathsf{Var}(\mathbf{E}_\tau) = \mathsf{Var}(\mathbf{C}_\tau) \times \mathsf{Var}(\mathbf{D}_\tau)$.

The category of all varieties VI

- **Theorem.** Let $\mathbf{C}_i$ be a minimal clone τ_i-algebra ($i = 1, 2$), $\mathbf{C}_1 \odot \mathbf{C}_2$ be the categorical product in $\mathcal{MCA}$ and ν be the type of $\mathbf{C}_1 \odot \mathbf{C}_2$. Then we have:

  1. The π_i-expansion $\mathbf{C}_i^{\pi_i}$ of $\mathbf{C}_i$ is a minimal clone ν-algebra, where π_i is the projection from $\mathbf{C}_1 \odot \mathbf{C}_2$ into $\mathbf{C}_i$ ($i = 1, 2$);
  2. $\mathbf{C}_i^{\pi_i}$ is purely isomorphic to $\mathbf{C}_i$ ($i = 1, 2$);
  3. $\mathbf{C}_1^{\pi_1} \times \mathbf{C}_2^{\pi_2} = \mathbf{C}_1 \odot \mathbf{C}_2$, where the product $\mathbf{C}_1^{\pi_1} \times \mathbf{C}_2^{\pi_2}$ is taken in the variety CA_ν;
  4. The varieties $\mathsf{Var}\,(\mathbf{C}_1^{\pi_1})_\nu$ and $\mathsf{Var}\,(\mathbf{C}_2^{\pi_2})_\nu$ are independent;
  5. $\mathsf{Var}\,(\mathbf{C}_1^{\pi_1} \times \mathbf{C}_2^{\pi_2})_\nu = \mathsf{Var}\,(\mathbf{C}_1^{\pi_1})_\nu \times \mathsf{Var}\,(\mathbf{C}_2^{\pi_2})_\nu = \mathsf{Var}\,(\mathbf{C}_1)_{\tau_1} \odot \mathsf{Var}\,(\mathbf{C}_2)_{\tau_2}$.

Future Work

- The focus of this work was on the representation theorems and their meaning for the theory of clones and ω-clones, and partly on the categorical aspects of clone algebras.

- Potential implications of clone algebras to universal algebra is deferred to future work that is currently in progress.

- What is the relationship between a variety $\mathcal{V}$ of type τ and the variety $\mathsf{Var}\,\mathbf{Cl}(\mathbf{F}_{\mathcal{V}})$ of clone τ-algebras generated by $\mathbf{Cl}(\mathbf{F}_{\mathcal{V}})$?

- To study the relationship between a variety $\mathcal{V}$ of pure clone algebras and the full subcategory of $\mathcal{VAR}$ constituted by all variety $\mathcal{W}$ such that the clone $\mathcal{W}$-algebra $\mathbf{Cl}(\mathbf{F}_{\mathcal{W}})$ is an element of $\mathcal{V}$.

- To understand whether some classical concepts of the theory of clones have a more general and algebraic formulation within the theory of clone algebras.

- Thank you!