A priori error estimates of regularized elliptic problems

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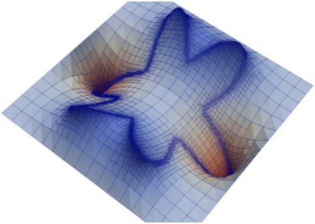
Backgrounds

Non-matching methods



Singular forcing term

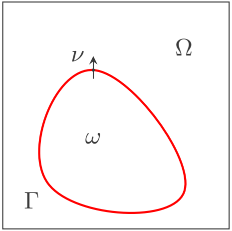
$$\int_B fv = \int_\Omega \int_B \delta(x-y) f(y) v(x) \, \mathrm{d}y \, \mathrm{d}x.$$

An interface problem

- Ω: bounded Lipschitz domain in $\mathbb{R}^d$ ($d = 2$ or 3);
- $\Gamma \subset \Omega$: a closed Lipschitz interface with co-dimension one;
- Γ is away from $\partial\Omega$, i.e.
$$\operatorname{dist}(\Gamma, \partial\Omega) > c_0;$$
- $f \in L^2(\Gamma)$.

Model problem

$$\begin{aligned} -\Delta u &= 0, &&\text{in } \Omega\backslash\Gamma, \\ [\![u]\!] &= 0, &&\text{on } \Gamma, \\ \left[\!\!\left[\frac{\partial u}{\partial \nu}\right]\!\!\right] &= f, &&\text{on } \Gamma, \\ u &= 0, &&\text{on } \partial\Omega. \end{aligned}$$



Weak formulation

Find $u \in V := H_0^1(\Omega)$ satisfying

$$\mathcal{A}(u, v) = \int_\Gamma f v \,\mathrm{d}\sigma := \langle F, v \rangle_{V', V}, \qquad \text{for all } v \in V.$$

Here

- $\mathcal{A}$ is the Dirichlet form,

$$\mathcal{A}(v, w) = \int_\Omega \nabla v^\intercal \nabla w \,\mathrm{d}x, \quad \text{for all } v, w \in V.$$

- The forcing data

$$F = \mathcal{M} f := \int_\Gamma \delta(x - y) f(y) \,\mathrm{d}\sigma_y,$$

with δ denoting the d-dimensional Dirac delta distribution.

Dirac delta approximation

Given $k \in \mathbb{N}$, let $\psi(x)$ in $L^\infty(\mathbb{R}^d)$ such that

1. **Compact supported:**

$$\mathit{supp}(\psi) \subset B_{r_0}(0)$$

2. k**-th order moments condition**:

$$\int_{\mathbb{R}^d} y_i^\alpha \psi(x-y)\,\mathrm{d}y = x_i^\alpha \qquad i = 1\ldots d, \quad 0 \le \alpha \le k, \qquad \text{for all } x \in \mathbb{R}^d;$$

Define the Dirac delta approximation

$$\delta^\varepsilon := \frac{1}{\varepsilon^d}\psi\left(\frac{x}{\varepsilon}\right)$$

Dirac delta approximation

L^1 growth control

$$\||x|^m \delta^\varepsilon(x)\|_{L^1(\mathbb{R}^d)} \preceq \varepsilon^m, \qquad 0 \le m \in \mathbb{R}.$$

Examples of ψ in $1d$:

- C^1: $\psi(x) = (1+\cos(\pi x))\chi_{(-1,1)}(x)/2$;
- C^∞: $\psi(x) = e^{1-1/(1-x^2)}\chi_{(-1,1)}(x)$;
- L^∞: $\psi(x) = \frac{1}{2}\chi_{(-1,1)}(x)$;
- *Polynomial class* $\psi^{k,s}$: [Tornberg, 2002].

Generating ψ from $\mathbb{R}$ to $\mathbb{R}^d$:

- *Radially symmetric:* ψ_ρ is supported in $[0,1]$ and set $\psi(x) := I_d\psi_\rho(|x|)$.
- *Tensor product:* $\psi(x) := \prod_{i=1}^d \psi_{1d}(x_i)$, for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$.

Data regularization/mollification

Regularization of $L^1(\Omega)$ functions

$$v^\varepsilon(x) := \int_\Omega \delta^\varepsilon(x-y)v(y)\,\mathrm{d}y, \qquad \text{for all } x \in \Omega,$$

Regularization of functionals in negative Sobolev spaces

$$\langle F^\varepsilon, v\rangle_{H^{-s}(\Omega),H^s(\Omega)} := \langle F, v^\varepsilon\rangle_{H^{-s}(\Omega),H^s(\Omega)}.$$

Regularized data for the interface problem

For $F = \mathcal{M}f := \int_\Gamma \delta(x-y)f(y)\,\mathrm{d}\sigma_y$, then

$$F^\varepsilon(x) = \int_\Gamma f(y)\delta^\varepsilon(y-x)\,\mathrm{d}y.$$

If $\psi(-x) = \psi(x)$, $F^\varepsilon(x) = \int_\Gamma f(y)\delta^\varepsilon(x-y)\,\mathrm{d}y.$

Regularized formulation and its FE approximation

Regularized problem

Find $\mathrm{u}^\varepsilon \in V$ satisfying

$$\mathcal{A}(\mathrm{u}^\varepsilon, v) = \langle F^\varepsilon, v\rangle_{V',V}, \qquad \text{for all } v \in V.$$

Finite element approximation

Finite element space $\mathbb{V}_h$:

- subordinate to a quasi-uniform mesh with the mesh size h;
- $\mathbb{V}_h \subset \mathbb{V}$;
- set of continuous piecewise linear functions.

Find $\mathrm{u}_h^\varepsilon \in \mathbb{V}_h$ satisfying

$$\mathcal{A}(\mathrm{u}_h^\varepsilon, v_h) = \langle F^\varepsilon, v_h\rangle_{V',V}, \qquad \text{for all } v_h \in \mathbb{V}_h.$$

Strang's Lemma

$$\|u - \mathbf{u}_h^\varepsilon\|_{H^1(\Omega)} \preceq \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_{H^1(\Omega)} + \sup_{w_h \in \mathbb{V}_h} \frac{\langle F - F^\varepsilon, w_h \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}}{\|w_h\|_{H_0^1(\Omega)}}$$

Regularity

For $f \in L^2(\Gamma)$, $F \in H^{s-1}(\Omega)$ with $0 \leq s < 1/2$.

$$\left| \int_\Gamma f(y)v(y)\,\mathrm{d}\sigma_y \right| \leq \|f\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)}$$
$$\preceq \|f\|_{L^2(\Gamma)} \|v\|_{H^{1-s}(\omega)} \leq \|f\|_{L^2(\Gamma)} \|v\|_{H^{1-s}(\Omega)}.$$

So $u \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$ by elliptic regularity.

Estimate of the space approximation using Scott-Zhang interpolant

$$\inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_{H^1(\Omega)} \leq \|u - I_h u\|_{H^1(\Omega)} \preceq h^s \|u\|_{H^{1+s}(\Omega)}.$$

Data consistancy

For $v \in H^1(\Omega)$,

$$\langle F - F^\varepsilon, v\rangle = \langle F, v - v^\varepsilon\rangle \preceq \|f\|_{L^2(\Gamma)}\|v - v^\varepsilon\|_{H^{1-s}(\omega)}.$$

Proposition [L. Heltai & WL, 2020]

If δ^ε has k-th order moment condition, for $v \in H^{k+1}(\Omega)$,

$$\|v - v^\varepsilon\|_{H^{1-s}(\omega)} \preceq \varepsilon^{k+s}\|v\|_{H^{k+1}(\omega^{\varepsilon_0})}.$$

Here $\varepsilon < \varepsilon_0$ and

$$\omega^\varepsilon := \cup_{x\in\omega} B_\varepsilon(x).$$

Idea of the proof: Using Taylor expansion (with the moment condition), Young's inequality for convolution and the L^1 growth control.

Error estimates

$$\|u - \mathbf{u}_h^\varepsilon\|_{H^1(\Omega)} \preceq \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_{H^1(\Omega)} + \sup_{w_h \in \mathbb{V}_h} \frac{\langle F - F^\varepsilon, w_h \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}}{\|w_h\|_{H_0^1(\Omega)}}$$

Theorem [L. Heltai & WL, 2020]

If δ^ε has the 0th order moment condition (i.e. $\int_{\mathbb{R}} \delta^\varepsilon = 1$),

$$\|u - \mathbf{u}^\varepsilon\|_{H^1(\Omega)} \preceq \varepsilon^s \|f\|_{L^2(\Gamma)}$$

and

$$\|u - \mathbf{u}_h^\varepsilon\|_{H^1(\Omega)} \preceq (h^s + \varepsilon^s)\|f\|_{L^2(\Gamma)},$$

where $s \in [0, \frac{1}{2})$.

L^2 error estimate for regularization

Theorem [L. Heltai & WL, 2020]

If δ^ε has the first order moment condition,

$$\|u - \mathbf{u}^\varepsilon\|_{L^2(\Omega)} \preceq \varepsilon^{s+1}\|f\|_{L^2(\Gamma)}.$$

Regularization estimates. If δ^ε has the first order moment condition,

$$\|v - v^\varepsilon\|_{H^{1-s}(\omega)} \preceq \varepsilon^{1+s}\|v\|_{H^2(\omega^{\varepsilon_0})}$$

H^2_{loc} regularity. Given $g \in L^2(\Omega)$, let $T : V' \to V$ be the solution operator satisfying

$$\mathcal{A}(Tg, v) = (g, v)_{L^2}, \qquad \text{for all } v \in V.$$

Then,

$$\|Tg\|_{H^2(\omega^{\varepsilon_0})} \preceq \|g\|_{L^2(\Omega)}.$$

Proof: a duality argument

The dual problem: find $z \in V$ such that

$$\mathcal{A}(v, z) = (u - \mathtt{u}^{\varepsilon}, v)_{\Omega}, \qquad \text{for all } v \in V.$$

Hence, we choose $v = u - \mathtt{u}^{\varepsilon}$ and obtain that

$$\begin{aligned} \|u - \mathtt{u}^{\varepsilon}\|^2_{L^2(\Omega)} &= \mathcal{A}(z, u - \mathtt{u}^{\varepsilon}) \\ &= \langle F - F^{\varepsilon}, z \rangle = \langle F, z - z^{\varepsilon} \rangle. \end{aligned} \tag{1}$$

Due to the interior regularity of z, $u - \mathtt{u}^{\varepsilon} \in H^1_0(\Omega) \subset L^2(\Omega)$ implies that

$$\|z\|_{H^2(\omega^{\varepsilon_0})} \preceq \|u - \mathtt{u}^{\varepsilon}\|_{L^2(\Omega)}.$$

We continue to estimate the right hand side of (1) by

$$\begin{aligned} \langle F, z - z^{\varepsilon} \rangle &\preceq \|f\|_{L^2(\Gamma)} \|z - z^{\varepsilon}\|_{H^{1-s}(\omega)} \\ &\preceq \varepsilon^{s+1} \|f\|_{L^2(\Gamma)} \|z\|_{H^2(\omega^{\varepsilon_0})} \\ &\preceq \varepsilon^{s+1} \|f\|_{L^2(\Gamma)} \|u - \mathtt{u}^{\varepsilon}\|_{L^2(\Omega)}, \end{aligned}$$

L^2 error estimate for the finite element approximation

Elliptic regularity for polygonal domain

$$\mathcal{A}(Tg, v) = \langle g, v \rangle_{V', V}, \qquad \text{for all } v \in V.$$

There exists $r \in (1/2, 1]$ and a positive constant C_r satisfying

$$\|Tg\|_{H^{1+r}(\Omega)} \leq C_r \|g\|_{H^{-1+r}(\Omega)}.$$

Theorem [L. Heltai & WL, 2020]

$$\|u - \mathrm{u}_h^\varepsilon\|_{L^2(\Omega)} \preceq (h^{r+s} + h^r \varepsilon^s + \varepsilon^{1+s}) \|f\|_{L^2(\Gamma)}.$$

If $h \sim \epsilon$ and $r = 1$,

$$\|u - \mathrm{u}_h^\varepsilon\|_{L^2(\Omega)} \preceq h^{3/2^-} \|f\|_{L^2(\Gamma)}.$$

Proof: Bound $\|u - \mathrm{u}^\varepsilon\|_{L^2}$ (previous theorem) and $\|\mathrm{u}^\varepsilon - \mathrm{u}_h^\varepsilon\|_{L^2}$ (duality argument) separately.

Test problem: square domain

- $\Gamma = \partial B_R(\mathbf{c})$ with $\mathbf{c} = (0.3, 0.3)^\intercal$ and $R = 0.2$;
- $f = \frac{1}{R}$ and non-homogeneous boundary condition $g = \ln(|x - \mathbf{c}|)$;
- The analytic solution:

$$u(x) = \begin{cases} -\ln(|x-\mathbf{c}|), & \text{if } |x-\mathbf{c}| > R, \\ -\ln(R), & \text{if } |x-\mathbf{c}| \le R. \end{cases}$$

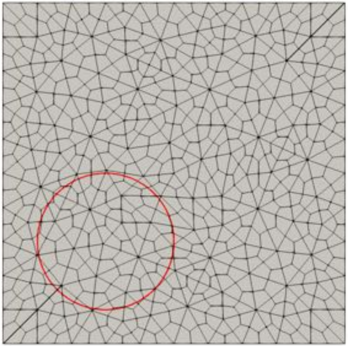
- Setting $\varepsilon = h$ yields

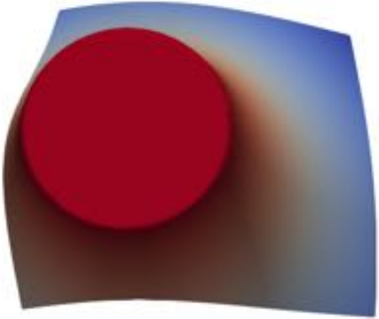
$$\|u - \mathbf{u}_h^\varepsilon\|_{H^1(\Omega)} \preceq h^{1/2} \sim \#\text{DoFs}^{-0.25},$$

and

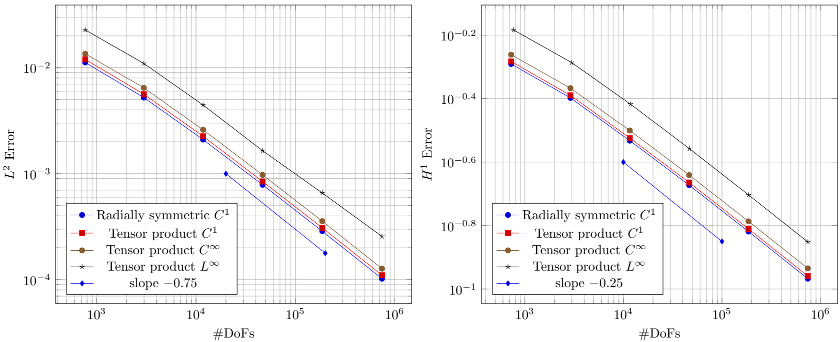
$$\|u - \mathbf{u}_h^\varepsilon\|_{L^2(\Omega)} \preceq h^{3/2} \sim \#\text{DoFs}^{-0.75}.$$

Test: coarse mesh and solution (744705 DoFs)





Test: L^2 and H^1 convergence tables



Test 2: unit cube

- $\Gamma = \partial B_R(\mathbf{c})$ with $\mathbf{c} = (0.3, 0.3, 0.3)^\intercal$ and $R = 0.2$;
- $f = \frac{1}{R^2}$ and nonhomogeneous boundary condition $g = 1/|x - \mathbf{c}|$;
- The analytic solution:

$$u(x) = \begin{cases} 1/|x-\mathbf{c}|, & \text{if } |x-\mathbf{c}| > R, \\ 1/R, & \text{if } |x-\mathbf{c}| \leq R. \end{cases}$$

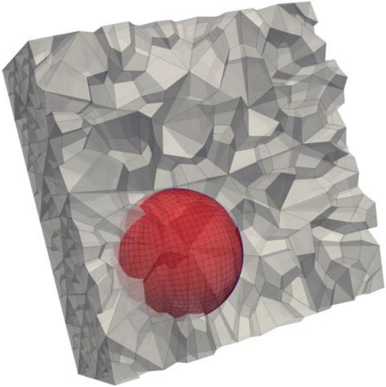
- Setting $\varepsilon = h$ yields

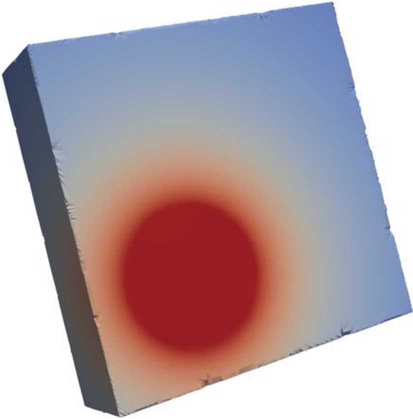
$$\|u - \mathbf{u}_h^\varepsilon\|_{H^1(\Omega)} \preceq h^{1/2} \sim \#\text{DoFs}^{-1/6},$$

and

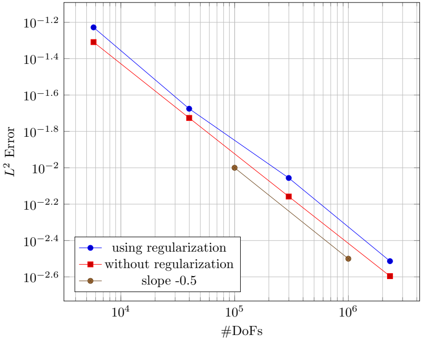
$$\|u - \mathbf{u}_h^\varepsilon\|_{L^2(\Omega)} \preceq h^{3/2} \sim \#\text{DoFs}^{-1/2}.$$

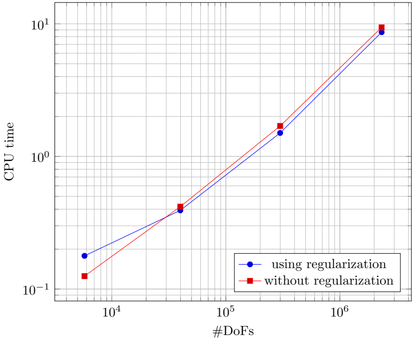
Test 2: coarse mesh of the unit cube and solution (2324113 DoFs)





Test 2: L^2 convergence table using regularization `tensorproduct` C^1 and without using regularization





Conclusion and Outlook

We have shown that the convergence rate of FEM for the regularized problem is the same as the direct approach if $\varepsilon \sim h$.

Outlook

- Adaptive finite element methods and its quasi-optimality (in preparation).
- Error analysis if the interface attaching the boundary, or even the corner.
- Fictitious domain methods using regularization.
- Time dependent problems.