Multifractal Eigenfunctions in a Singular Quantum Billiard

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$$r_2(n) = \#\{(x, y) \in \mathbb{Z}^2 \mid n = x^2 + y^2\}$$

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Consider a self-adjoint extension $-\Delta_\varphi$, $\varphi \in [-\pi, \pi)$, of $-\Delta$ restricted to the domain $C_c^\infty(\mathbb{T}^2 \setminus \{x_0\})$ – this operator corresponds to the **weak coupling quantization** for a square torus with a Dirac-delta potential.

The spectrum of the operator $-\Delta_\varphi$ consists of two parts:

(i) old (Laplace) eigenvalues with multiplicity reduced by 1

(ii) new eigenvalues with multiplicity 1 which interlace with the Laplace eigenvalues

The eigenspace associated with old eigenvalues is simply the codimension one subspace of Laplace eigenfunctions which vanish at x_0. The old eigenfunctions do not feel the presence of the potential. We will only be interested in the new eigenfunctions which *do* feel the presence of the potential.

The new eigenfunctions are Green functions associated with the resolvent of the Laplacian, where one variable has been fixed as the position of the singularity x_0. Because of translation invariance we may set $x_0 = 0$. We then have the following L^2-representation of the new eigenfunctions

$$G_\lambda(x) \stackrel{L^2}{=} \sum_{\xi \in \mathbb{Z}^2} \frac{e^{i\langle \xi, x\rangle}}{|\xi|^2 - \lambda} = \sum_{n \in \mathcal{N}} \frac{1}{n - \lambda} \sum_{|\xi|^2 = n} e^{i\langle \xi, x\rangle}.$$

The new eigenvalues may be determined as the solutions of the *spectral equation*

$$\sum_{n \in \mathcal{N}} r_2(n) \left\{ \frac{1}{n - \lambda} - \frac{n}{n^2 + 1} \right\} = c_0 \tan(\frac{\varphi}{2}).$$

The **strong coupling quantization** (c.f. Shigehara; Bogomolny, Gerland & Schmit; Kurlberg & Rudnick; Rudnick & Ueberschär; Kurlberg & Ueberschär; ...) corresponds to a logarithmic renormalization of the coupling parameter, $\tan(\varphi_\lambda/2) \sim c \log \lambda$, and is equivalent to a truncation of the sum on the l.h.s. of the spectral equation outside a window of size $\lambda^{1/2}$ [the tail of the sum has a logarithmic asymptotic as $\lambda \to +\infty$ that is cancelled by the leading term in the asymptotic of $\tan(\varphi_\lambda/2)$. The second order term in this asymptotic is related to local information about the position of the new eigenvalues.]

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It is similar in many respects to *quantum star graphs* (Berkolaiko & KKeating; Berkolaiko, Bogomolny & Keating; Berkolaiko, Keating & Winn; Keating, Marklof & Winn; ...).

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In the strong coupling case $\langle \Delta_j \rangle_x = (\log x)^{\alpha + o(1)}, \ \alpha = \alpha(\Lambda) \in (-\frac{1}{2}, \frac{1}{2}]$.

Multifractals

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Multifractals: geometrical objects characterised by a spectrum of scaling dimensions, for example where the local scaling dimension differs from point to point and is itself fractally distributed.

Denote by (X, μ) a measure space. In multifractal analysis one usually evaluates μ on a box partition $\mathcal{B}_r$ of X, where r denotes the side length of the boxes. To characterise the self-similarity of the measure μ one studies the behaviour of the moment sum

$$M_q(r) = \sum_{B \in \mathcal{B}_r} \mu(B)^q$$

in the limit as $r \to 0$. One expects a scaling law of the form $M_p(r) \sim r^{D_q}$, where D_q denotes the fractal exponent.

Consider the Laplace-Beltrami operator $-\Delta$ on a two-dimensional Euclidean compact manifold $\mathcal{M}$ with or without boundary. An eigenfunction ψ_λ of $-\Delta$, satisfying $(\Delta + \lambda)\psi_\lambda = 0$, may be represented as a superposition of plane waves

$$\psi_\lambda(x) = \sum_{\xi \in \mathcal{L}} \hat{\psi}_\lambda(\xi) e^{i\xi \cdot x}$$

by embedding the Euclidean billiard in a rectangular enclosure $\mathcal{R}$ and representing ψ_λ with respect to the canonical orthonormal basis of complex exponentials $e^{i\xi \cdot x}$, where $\xi \in \mathcal{L}$ and $\mathcal{L}$ is a rectangular lattice.

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Let us introduce the probability measure

$$\mu_\lambda(\xi) = |\hat{\psi}_\lambda(\xi)|^2,$$

where we assume that ψ is L^2-normalized, and consider the moment sum

$$M_q(\lambda) = \sum_{\xi \in \mathcal{L}} \mu_\lambda(\xi)^q.$$

The semiclassical regime: $\lambda \to +\infty$

To begin with, consider the case of a finite-dimensional Hilbert space. Each eigenstate may be represented by a vector $v \in \mathbb{C}^d$, where d denotes the dimension of the space. Let us normalize such that the largest coordinate equals 1 (ℓ^∞ normalization).

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The exponent D_q describes how the moment sum associated with v scales with respect to the dimension d:

$$\sum_{k=1}^{n} v_k^{2q} \sim d^{D_q}.$$

If v is flat, then all entries equal 1 and $D_q = 1$. If v is maximally localized, meaning that one entry equals 1 and all others vanish, then $D_q = 0$.

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The fractal exponent D_q may be defined in terms of the Renyi entropy of the probability measure $p_i = v_i^2/\|v\|^2$ in the following way:

$$D_q = \frac{H_q(v)}{\log d}$$

with

$$H_q(v) = \frac{1}{1-q} \log \left(\sum_{i=1}^{d} p_i^q \right).$$

In the case of Šeba's billiard we have the following explicit formula for the probability measure μ_λ on the lattice $\mathbb{Z}^2$:

$$\mu_\lambda(\xi) = (|\xi|^2 - \lambda)^{-2} \left(\sum_{\xi \in \mathbb{Z}^2} (|\xi|^2 - \lambda)^{-2} \right)^{-1} .$$

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Firstly, let us determine the scaling parameter. Since we are dealing with an infinite-dimensional space, the dimension must be replaced with the number of important components N in the moment sum. N will be defined in terms of the average number of lattice points in the annulus which supports most of the mass of the measure μ_λ; in particular N will depend on λ.

The moment sum takes the form

$$M_q(\lambda_j) = \frac{\zeta_{\lambda_j}(2q)}{\zeta_{\lambda_j}(2)^q}$$

where we introduce the zeta function

$$\zeta_\lambda(s) = \sum_{m \in \mathcal{N}} r_Q(m)(m-\lambda)^{-s}, \quad \mathrm{Re}\, s > 1,$$

$r_Q(n)$ denotes the representation number of the quadratic form defined by $Q(\xi_1, \xi_2) = |\xi|^2$ for $\xi = (\xi_1, \xi_2) \in \mathcal{L}$, and $\mathcal{N}$ denotes the set of numbers which is representable as a value of Q for some $\xi \in \mathcal{L}$.

In the following we will consider the *arithmetic case*, when $\mathcal{L} = \mathbb{Z}^2$ and $r_Q(n) = r_2(n)$.

Weak Coupling

The moment sum can be rewritten as

$$M_q(\lambda_j) = \frac{\zeta_{\lambda_j}(2q)}{\zeta_{\lambda_j}(2)^q} = \frac{m_q(\lambda_j)}{m_1(\lambda_j)^q}$$

where

$$m_q(\lambda_j) = r_2(\tilde{n}_j) + \Delta_j^{2q} \sum_{m \neq \tilde{n}_j} r_2(m)(m - \lambda_j)^{-2q}$$

and, because Δ_j is typically small, only the term $r_2(\tilde{n}_j)$ contributes in the numerator and denominator respectively. Hence, along a full density subsequence of eigenvalues,

$$M_q(\lambda_j) \sim r_2(\tilde{n}_j)^{1-q}.$$

For a full density subsequence of $n \in \mathcal{N}$ we have

$$r_2(n) = (\log n)^{\frac{1}{2}\log 2 + o(1)}$$

because $r_2(n)$ is close to its logarithmic normal order. Therefore, we define the scaling parameter as the logarithmic normal order of $r_2(\tilde{n}_j)$:

$$N = (\log \tilde{n}_j)^{\frac{1}{2}\log 2}$$

We define the semiclassical fractal exponent, associated with a sequence of eigenvalues, as follows.
Let $q > \frac{1}{2}$. Let Λ be a sequence of eigenvalues which accumulates at infinity. We define the **general entropy** associated with the eigenfunction ψ_λ as

$$h_q(\psi_\lambda) = \log m_q(\lambda).$$

For $q \neq 1$ the **Renyi entropy** associated with the probability measure μ_λ on $\mathbb{Z}^2$ may then be defined as

$$H_q(\mu_\lambda) = \frac{h_q(\psi_\lambda) - q h_1(\psi_\lambda)}{1-q} = \frac{1}{1-q} \log \left(\sum_{\xi \in \mathbb{Z}^2} \mu_\lambda(\xi)^q \right).$$

The fractal exponent associated with Λ is defined as

$$d_q^\Lambda = \limsup_{\lambda \in \Lambda \to +\infty} \frac{h_q(\psi_\lambda)}{\log N} \quad \text{with respect to the } \ell^\infty\text{-normalization,}$$

$$D_q^\Lambda = \limsup_{\lambda \in \Lambda \to +\infty} \frac{H_q(\mu_\lambda)}{\log N} \quad \text{with respect to the } \ell^2\text{-normalization.}$$

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Theorem

Let Λ_w denote the sequence of new eigenvalues in the weak coupling regime. There exists a full density subsequence $\Lambda' \subset \Lambda_w$ such that $d_q(\Lambda') = D_q(\Lambda') = 1$.

Strong Coupling

To determine the scaling parameter N we again have to calculate the number of important terms in the moment sum. This is closely related to the number of lattice points in an annulus

$$A(\lambda, G) = \{v \in \mathbb{R}^2 \mid ||\xi|^2 - \lambda| \leq G\},$$

where $G = G(\lambda)$ is just large enough such that the terms corresponding to lattice points outside the annulus $A(\lambda, G)$ are negligible, when $\lambda \to +\infty$. In particular, G must grow faster than the mean spacing of the Laplace eigenvalues: $G/\sqrt{\log \lambda} \to +\infty$.

Theorem

Let Λ_s denote a sequence of new eigenvalues such that $\alpha = \alpha(\Lambda_s) \in (\frac{1}{4}, \frac{1}{2})$. There exists a full density subsequence $\Lambda' \subset \Lambda_s$ such that for any $q > 1$ which satisfies $(1 - \log 2)(2 - 4\alpha)^{-1} < q \leq (2 - 4\alpha)^{-1}$ we have the following formulae, which may be analytically continued to the complex plane,

$$d_q(\Lambda') = \frac{1}{2\alpha}\left(1 - \frac{1}{2q}\right)\log 2$$

and for a constant $c \in [\frac{1}{2}\log 2, 1]$

$$D_q(\Lambda') = \frac{1}{2\alpha}\left(1 - \frac{1}{2q}\right)\frac{2cq - \log 2}{q - 1}.$$

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This leads to the study of the modified moment sums

$$M_q^*(\lambda) = \zeta_\lambda^*(2q)$$

where

$$\zeta_\lambda^*(s) = \sum_{n \in \mathcal{N} \setminus \{0\}} r_Q(n)(n-\lambda)^{-s}, \quad \mathrm{Re}\, s > 1.$$

The parameter N does not depend on λ in this regime and, therefore, we remove it from the definition of the fractal exponent. Let $q > 1$. We define the fractal exponent as follows

$$d_q^* = \lim_{\lambda \to 0} \log \zeta_\lambda^*(2q)$$

and

$$D_q^* = \frac{d_q^* - q d_1^*}{1 - q}$$

We denote Epstein's zeta function by

$$\zeta_Q(s) = \sum_{(m,n) \neq (0,0)} Q(m,n)^{-s}, \quad \operatorname{Re} s > 1$$

which is known to satisfy the functional equation

$$\zeta_Q(s) = \varphi_Q(s) \zeta_Q(1-s).$$

Theorem

The fractal exponent D_q^* admits an analytic continuation to the full complex plane in q. It satisfies the following symmetry relation with respect to the parameter value $q = \frac{1}{4}$:

$$D_{1/2-q}^* = \frac{1-q}{\frac{1}{2}+q}\left(D_q^* + \frac{\log \varphi_Q(2q) + (2q - \frac{1}{2})\log \zeta_Q(2)}{1-q}\right).$$

In particular, in the limit as $q \to 1$, the fractal exponent converges to the Shannon entropy of the measure $\mu_\lambda(\xi) = \zeta_Q(2)^{-1}|\xi|^{-4}$:

$$\lim_{q\to 1} D_q^* = \log \zeta_Q(2) - 2\frac{\zeta_Q'(2)}{\zeta_Q(2)}$$

which we may rewrite as

$$H_{Sh.}(\mu_\lambda) = -\sum_{\xi\in\mathbb{Z}^2} \frac{1}{\zeta_Q(2)|\xi|^4}\log\left(\frac{1}{\zeta_Q(2)|\xi|^4}\right).$$

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summary

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- mutifractality of semiclassical eigenfunctions in arithmetic Šeba billiards in the strong coupling quantization, with a formula for the scaling exponents D_q,
- and mutifractality in the ground state, where we can prove symmetry in D_q around $q = 1/4$ as a consequence of the functional equation of the Epstein zeta-function.